22. VECTORS AND THEIR PROPERTIES

In our daily life, we generally come across two types of quantities, namely scalars and vectors.

SCALARS *A quantity that has magnitude only is known as a scalar.*

Examples Each of the quantities *mass, length, time, temperature, density, speed,* etc., is a scalar.

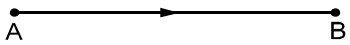
VECTORS *A quantity that has magnitude as well as direction is called a vector.*

Examples Each of the quantities *force, velocity, acceleration* and *momentum* is a vector.

However, we define a vector as given below.

'A directed line segment is called a vector.'

A directed line segment with ***initial point*** A and the ***terminal point*** B, is the vector denoted by $\overrightarrow{AB}$.



The magnitude of $\overrightarrow{AB}$ is denoted by $|\overrightarrow{AB}|$.

REMARK We usually denote a vector by a single letter with an arrow on it and its magnitude is denoted by this letter only.

Thus, $\overrightarrow{AB} = \vec{a}$ and $|\overrightarrow{AB}| = |\vec{a}| = a$.

UNIT VECTOR *A vector* $\vec{a}$ *is called a unit vector if* $|\vec{a}| = 1$ *and it is denoted by* $\hat{a}$.

EQUAL VECTORS *Two vectors* $\vec{a}$ *and* $\vec{b}$ *are said to be equal if they have the same magnitude and the same direction regardless of the positions of their initial points.*

NEGATIVE OF A VECTOR *A vector having the same magnitude as that of a given vector* $\vec{a}$ *and the direction opposite to that of* $\vec{a}$ *is called the negative of* $\vec{a}$, *to be denoted by* $-\vec{a}$.

Thus, if $\overrightarrow{AB} = \vec{a}$ then $\overrightarrow{BA} = -\vec{a}$.

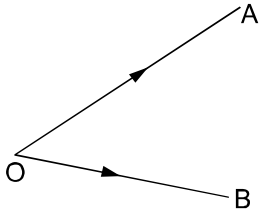
ZERO OR NULL VECTOR *A vector whose initial and terminal points coincide is called a zero vector, denoted by* $\vec{0}$.

Clearly, the magnitude of a zero vector is 0 but it cannot be assigned a definite direction.

Thus, $\overrightarrow{AA} = \vec{0}$.

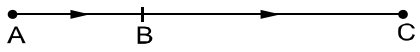
COINITIAL VECTORS *Two or more vectors having the same initial point are called coinitial vectors.*

In the given figure, $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are the two coinitial vectors having the same initial point O.



COLLINEAR VECTORS *Vectors having the same or parallel supports are known as collinear vectors.*

In the given figure $\overrightarrow{AB}$, $\overrightarrow{BC}$ and $\overrightarrow{AC}$ are collinear vectors.



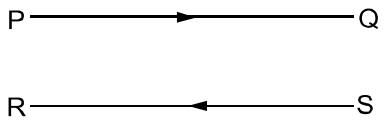
LIKE VECTORS *Collinear vectors having the same direction are called like vectors.*

Thus, $\overrightarrow{AB}$, $\overrightarrow{BC}$ and $\overrightarrow{AC}$ shown above are like vectors.

UNLIKE VECTORS *Collinear vectors having opposite directions are known as unlike vectors.*

In the given figure, $PQ \,||\, RS$.

$\therefore$ $\overrightarrow{PQ}$ and $\overrightarrow{SR}$ are unlike vectors.



FREE VECTORS *If the initial point of a vector is not specified then it is said to be a free vector.*

LOCALISED VECTORS *A vector drawn parallel to a given vector through a specified point as the initial point is called a localised vector.*

COPLANAR VECTORS *Three or more nonzero vectors lying in the same plane or parallel to the same plane are said to be coplanar, otherwise they are called noncoplanar.*

POSITION VECTOR OF A POINT *Let O be the origin and let A be a point such that $\overrightarrow{OA} = \vec{a}$, then we say that the position vector of A is $\vec{a}$.*

SOLVED EXAMPLES

EXAMPLE 1 *Classify the following measures as scalars and vectors:*

(i) 40 seconds *(ii)* 100 m^2 *(iii)* 30 gm/cm^3

(iv) 60 km/hr *(v)* 56 m/s *towards south*

SOLUTION (i) 40 seconds represents time, which is scalar.

(ii) 100 m^2 represents an area, which is scalar.

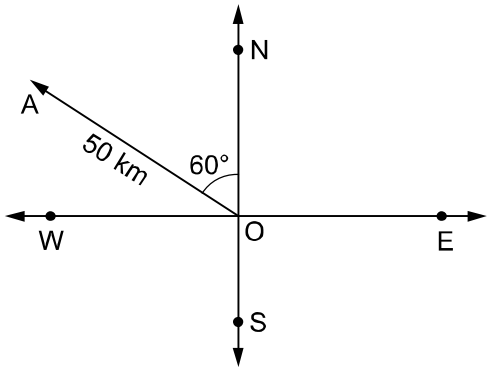
(iii) 30 gm/cm^3 represents density, which is scalar.

(iv) 60 km/hr represents speed, which is scalar.

(v) 56 m/s towards south represents velocity, which is a vector.

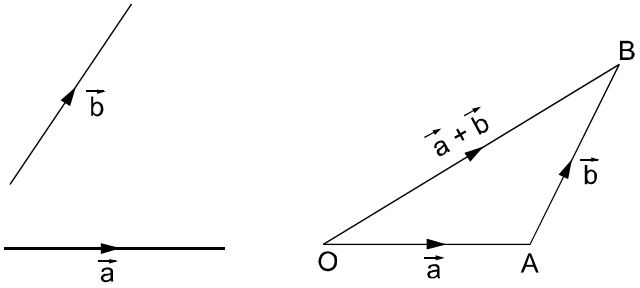
EXAMPLE 2 *Represent graphically a displacement of* 50 km, 60° *west of north.*

SOLUTION The vector $\overrightarrow{OA}$ given below represents a displacement of 50 km, 60° west of north.



VECTOR ADDITION

Let $\vec{a}$ and $\vec{b}$ be any two vectors. Take any point O and draw segments $\overrightarrow{OA}$ and $\overrightarrow{AB}$ such that $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{AB} = \vec{b}$. Join OB. Then, $\overrightarrow{OB}$ is called the ***sum,*** or ***resultant*** of $\vec{a}$ and $\vec{b}$.

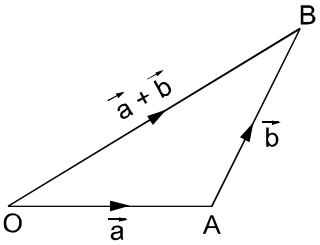


$\therefore \quad (\vec{a} + \vec{b}) = \overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}.$

TRIANGLE LAW OF ADDITION OF VECTORS

In a $\triangle OAB$, if $\overrightarrow{OA}$ and $\overrightarrow{AB}$ represent $\vec{a}$ and $\vec{b}$ respectively, then $\overrightarrow{OB}$ represents $(\vec{a} + \vec{b})$.

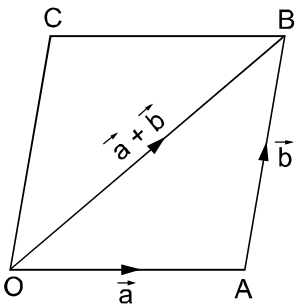
This is known as *Triangle Law of Addition of Vectors.*



PARALLELOGRAM LAW OF ADDITION OF VECTORS

In a ||gm OABC, if $\overrightarrow{OA}$ and $\overrightarrow{AB}$ represent $\vec{a}$ and $\vec{b}$ respectively, then $\overrightarrow{OB}$ represents $(\vec{a} + \vec{b})$.

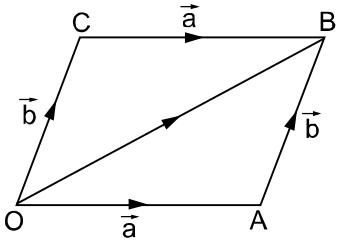
This is known as *Parallelogram Law of Addition of Vectors.*



Laws of Addition of Vectors

THEOREM 1 (Commutative Law) *Vector addition is commutative,*
i.e., $\vec{a}+\vec{b}=\vec{b}+\vec{a}$.

PROOF Let $\vec{a}$ and $\vec{b}$ be the given vectors represented by $\overrightarrow{OA}$ and $\overrightarrow{AB}$ respectively. Complete the parallelogram $OABC$.



Then, $\overrightarrow{OC}=\overrightarrow{AB}=\vec{b}$

and $\overrightarrow{CB}=\overrightarrow{OA}=\vec{a}$.

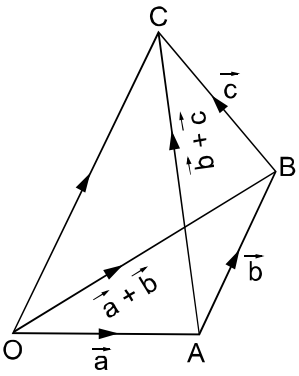
$\therefore \quad \vec{a}+\vec{b}=\overrightarrow{OA}+\overrightarrow{AB}=\overrightarrow{OB}$

and $\vec{b}+\vec{a}=\overrightarrow{OC}+\overrightarrow{CB}=\overrightarrow{OB}$.

Hence, $\vec{a}+\vec{b}=\vec{b}+\vec{a}$.

THEOREM 2 (Associative Law) *Vector addition is associative,*
i.e., $(\vec{a}+\vec{b})+\vec{c}=\vec{a}+(\vec{b}+\vec{c})$.

PROOF Let $\overrightarrow{OA}=\vec{a}$, $\overrightarrow{AB}=\vec{b}$ and $\overrightarrow{BC}=\vec{c}$.



Join OB, OC and AC.

$$(\vec{a}+\vec{b})+\vec{c}=(\overrightarrow{OA}+\overrightarrow{AB})+\overrightarrow{BC}$$
$$=(\overrightarrow{OB}+\overrightarrow{BC}) \quad [\because\ \overrightarrow{OA}+\overrightarrow{AB}=\overrightarrow{OB}]$$
$$=\overrightarrow{OC}$$

$$\vec{a}+(\vec{b}+\vec{c})=\overrightarrow{OA}+(\overrightarrow{AB}+\overrightarrow{BC})$$
$$=\overrightarrow{OA}+\overrightarrow{AC} \quad [\because\ \overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC}]$$
$$=\overrightarrow{OC}$$

$\therefore \quad (\vec{a}+\vec{b})+\vec{c}=\vec{a}+(\vec{b}+\vec{c})$.

THEOREM 3 (Existence of Additive Identity) *For any vector* $\vec{a}$, *prove that*
$\vec{a}+\vec{0}=\vec{0}+\vec{a}=\vec{a}$.

PROOF Let $\overrightarrow{OA}=\vec{a}$. Then, $\vec{a}+\vec{0}=\overrightarrow{OA}+\overrightarrow{AA}=\overrightarrow{OA}=\vec{a}$

and, $\vec{0}+\vec{a}=\overrightarrow{OO}+\overrightarrow{OA}=\overrightarrow{OA}=\vec{a}$.

$\therefore \quad \vec{a}+\vec{0}=\vec{0}+\vec{a}=\vec{a}$.

REMARK The vector $\vec{0}$ is called the *additive identity* for vectors.

THEOREM 4 (Existence of Additive Inverse) *For any vector* $\vec{a}$, *prove that* $\vec{a} + (-\vec{a}) = (-\vec{a}) + \vec{a} = \vec{0}$.

PROOF Let $\overrightarrow{OA} = \vec{a}$. Then, $\overrightarrow{AO} = -\vec{a}$.

$\therefore \quad \vec{a} + (-\vec{a}) = \overrightarrow{OA} + \overrightarrow{AO} = \overrightarrow{OO} = \vec{0}$

and, $(-\vec{a}) + \vec{a} = \overrightarrow{AO} + \overrightarrow{OA} = \overrightarrow{AA} = \vec{0}$.

Hence, $\vec{a} + (-\vec{a}) = (-\vec{a}) + \vec{a} = \vec{0}$.

REMARK The vector $-\vec{a}$ is called the *additive inverse* of $\vec{a}$.

DIFFERENCE OF TWO VECTORS *For any two vectors* $\vec{a}$ *and* $\vec{b}$, *we define*

$\vec{a} - \vec{b} = \vec{a} + (-\vec{b})$.

Let $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{OB} = \vec{b}$. Then, $\overrightarrow{BO} = -\vec{b}$.

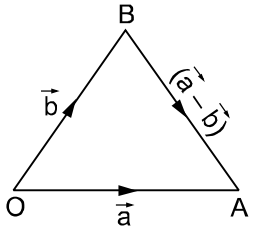
$\therefore \quad (\vec{a} - \vec{b}) = \vec{a} + (-\vec{b})$

$= \overrightarrow{OA} + \overrightarrow{BO} = \overrightarrow{BO} + \overrightarrow{OA} = \overrightarrow{BA}$

$\therefore \quad (\overrightarrow{OA} - \overrightarrow{OB}) = \overrightarrow{BA}$.

In a similar way, we can prove that

$(\overrightarrow{OB} - \overrightarrow{OA}) = \overrightarrow{AB}$.



AN IMPORTANT REMARK:

(i) $\overrightarrow{AB}$ = (*position vector of B*) – (*position vector of A*)

(ii) $\overrightarrow{BA}$ = (*position* vector of *A*) – (*position vector of B*)

Scalar Multiplication of a Vector

The scalar multiple of $\vec{a}$ *by a scalar k is the vector* $k\vec{a}$ *such that*

(i) $|k\vec{a}| = |k||\vec{a}|$,

(ii) direction of $k\vec{a}$ is the same as that of $\vec{a}$, when $k > 0$ and opposite to that of $\vec{a}$ when $k < 0$.

Examples (i) $5\vec{a}$ is the vector whose magnitude is 5 times the magnitude of $\vec{a}$ and whose direction is the same as that of $\vec{a}$.

(ii) $-2\vec{a}$ is the vector whose magnitude is 2 times the magnitude of $\vec{a}$ and whose direction is opposite to that of $\vec{a}$.

Components of a Vector

Let O be the origin and let $P(x, y, z)$ be any point in space. Let $\hat{i}, \hat{j}, \hat{k}$ be unit vectors along the x-axis, y-axis and z-axis respectively. Let the position vector of P be $\vec{r}$.

Then, $\vec{r} = (x\hat{i} + y\hat{j} + z\hat{k})$.

This form of a vector is called its ***component form***.

Here x, y, z are called the ***scalar components*** of $\vec{r}$ and $x\hat{i}, y\hat{j}, z\hat{k}$ are called its ***vector components.***

Also, $|\vec{r}| = |x\hat{i} + y\hat{j} + z\hat{k}| = \sqrt{x^2 + y^2 + z^2}$.

Direction Ratios and Direction Cosines of a Vector

Consider a vector $\vec{r} = a\hat{i} + b\hat{j} + c\hat{k}$.

Then, the numbers a, b, c are called the ***direction ratios*** of $\vec{r}$.

Direction cosines of $\vec{r}$ are given by

$$\frac{a}{\sqrt{a^2 + b^2 + c^2}}, \quad \frac{b}{\sqrt{a^2 + b^2 + c^2}} \quad \text{and} \quad \frac{c}{\sqrt{a^2 + b^2 + c^2}}.$$

NOTE If l, m, n are the direction cosines of a vector then we always have $(l^2 + m^2 + n^2) = 1$.

REMARK If $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ be any two points in space then direction ratios of $\overrightarrow{AB}$ are $(x_2 - x_1), (y_2 - y_1), (z_2 - z_1)$ and direction cosines of $\overrightarrow{AB}$ are

$$\frac{(x_2 - x_1)}{r}, \quad \frac{(y_2 - y_1)}{r}, \quad \frac{(z_2 - z_1)}{r},$$

where $r = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$.

SOLVED EXAMPLES

EXAMPLE 1 *Let* $\vec{a} = a_1\hat{i} + 3\hat{j} + a_3\hat{k}$ *and* $\vec{b} = 2\hat{i} + b_2\hat{j} + \hat{k}$. *If* $\vec{a} = \vec{b}$, *find the values of* a_1, b_2 *and* a_3.

SOLUTION $\vec{a} = \vec{b} \Leftrightarrow a_1\hat{i} + 3\hat{j} + a_3\hat{k} = 2\hat{i} + b_2\hat{j} + \hat{k}$

$\Leftrightarrow a_1 = 2, \; b_2 = 3, \; a_3 = 1.$

EXAMPLE 2 *Let* $\vec{a} = 3\hat{i} + 2\hat{j}$ *and* $\vec{b} = 2\hat{i} + 3\hat{j}$. *Is* $|\vec{a}| = |\vec{b}|$? *Is* $\vec{a} = \vec{b}$?

SOLUTION We have

$$|\vec{a}| = \sqrt{3^2 + 2^2} = \sqrt{13} \text{ and } |\vec{b}| = \sqrt{2^2 + 3^2} = \sqrt{13}.$$

$\therefore \quad |\vec{a}| = |\vec{b}|.$

But, $3\hat{i} + 2\hat{j} \neq 2\hat{i} + 3\hat{j}$ and therefore, $\vec{a} \neq \vec{b}$.

EXAMPLE 3 *Find a unit vector in the direction of the vector* $\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$.

SOLUTION $\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k} \Rightarrow |\vec{a}| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}.$

Unit vector in the direction of $\vec{a}$ is given by

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{(\hat{i} + 2\hat{j} + 3\hat{k})}{\sqrt{14}} = \left(\frac{1}{\sqrt{14}}\hat{i} + \frac{2}{\sqrt{14}}\hat{j} + \frac{3}{\sqrt{14}}\hat{k}\right).$$

EXAMPLE 4 *Write a vector of magnitude* 15 *units in the direction of the vector* $(\hat{i} - 2\hat{j} + 2\hat{k})$. **[CBSE 2010]**

SOLUTION Let $\vec{a} = (\hat{i} - 2\hat{j} + 2\hat{k})$. Then,

a unit vector in the direction of $\vec{a}$ is given by

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{(\hat{i} - 2\hat{j} + 2\hat{k})}{\sqrt{1^2 + (-2)^2 + 2^2}} = \frac{1}{3}(\hat{i} - 2\hat{j} + 2\hat{k}).$$

$\therefore$ a vector of magnitude 15 in the direction of $\vec{a}$

$$= 15 \times \frac{1}{3}(\hat{i} - 2\hat{j} + 2\hat{k}) = (5\hat{i} - 10\hat{j} + 10\hat{k}).$$

EXAMPLE 5 *Find a unit vector parallel to the sum of the vectors* $(\hat{i} + \hat{j} + \hat{k})$ *and* $(2\hat{i} - 3\hat{j} + 5\hat{k})$. **[CBSE 2012]**

SOLUTION Let $\vec{a} = (\hat{i} + \hat{j} + \hat{k})$ and $\vec{b} = (2\hat{i} - 3\hat{j} + 5\hat{k})$. Then,

$$(\vec{a} + \vec{b}) = (\hat{i} + \hat{j} + \hat{k}) + (2\hat{i} - 3\hat{j} + 5\hat{k}) = (3\hat{i} - 2\hat{j} + 6\hat{k}).$$

Required unit vectors parallel to $(\vec{a} + \vec{b})$ are

$$\pm\frac{(3\hat{i} - 2\hat{j} + 6\hat{k})}{\sqrt{3^2 + (-2)^2 + 6^2}} = \pm\frac{1}{7}(3\hat{i} - 2\hat{j} + 6\hat{k}).$$

EXAMPLE 6 *If* $\vec{a}=(\hat{i}+\hat{j}+\hat{k}),\ \vec{b}=(4\hat{i}-2\hat{j}+3\hat{k})$ *and* $\vec{c}=(\hat{i}-2\hat{j}+\hat{k})$, *find a vector of magnitude 6 units which is parallel to the vector* $(2\vec{a}-\vec{b}+3\vec{c})$. **[CBSE 2011C]**

SOLUTION We have

$$(2\vec{a}-\vec{b}+3\vec{c})=2(\hat{i}+\hat{j}+\hat{k})-(4\hat{i}-2\hat{j}+3\hat{k})+3(\hat{i}-2\hat{j}+\hat{k})$$
$$=(2-4+3)\hat{i}+(2+2-6)\hat{j}+(2-3+3)\hat{k}$$
$$=(\hat{i}-2\hat{j}+2\hat{k}).$$

Unit vectors parallel to $(2\vec{a}-\vec{b}+3\vec{c})$ are

$$\pm\frac{(\hat{i}-2\hat{j}+2\hat{k})}{\sqrt{1^2+(-2)^2+2^2}}=\pm\frac{1}{3}(\hat{i}-2\hat{j}+2\hat{k}).$$

Required vectors of magnitude 6 units are

$$\pm\left\{6\times\frac{1}{3}(\hat{i}-2\hat{j}+2\hat{k})\right\}=\pm 2(\hat{i}-2\hat{j}+2\hat{k}).$$

EXAMPLE 7 *Find a unit vector in the direction of* $\overrightarrow{AB}$, *where* $A(1, 2, 3)$ *and* $B(4, 5, 6)$ *are the given points.*

SOLUTION We have

p.v. of $A=(\hat{i}+2\hat{j}+3\hat{k})$ and p.v. of $B=(4\hat{i}+5\hat{j}+6\hat{k})$.

$\therefore$ $\overrightarrow{AB}=$ (p.v. of B) $-$ (p.v. of A)

$$=(4\hat{i}+5\hat{j}+6\hat{k})-(\hat{i}+2\hat{j}+3\hat{k})=(3\hat{i}+3\hat{j}+3\hat{k}),\text{ and}$$

$$|\overrightarrow{AB}|=\sqrt{3^2+3^2+3^2}=\sqrt{27}.$$

$\therefore$ unit vector in the direction of $\overrightarrow{AB}$

$$=\frac{\overrightarrow{AB}}{|\overrightarrow{AB}|}=\frac{(3\hat{i}+3\hat{j}+3\hat{k})}{\sqrt{27}}=\frac{3(\hat{i}+\hat{j}+\hat{k})}{3\sqrt{3}}=\frac{(\hat{i}+\hat{j}+\hat{k})}{\sqrt{3}}$$
$$=\left(\frac{1}{\sqrt{3}}\hat{i}+\frac{1}{\sqrt{3}}\hat{j}+\frac{1}{\sqrt{3}}\hat{k}\right).$$

EXAMPLE 8 *For what value of a, the vectors* $(2\hat{i}-3\hat{j}+4\hat{k})$ *and* $(a\hat{i}+6\hat{j}-8\hat{k})$ *collinear?* **[CBSE 2011]**

SOLUTION The given vectors are collinear only when

$$(a\hat{i}+6\hat{j}-8\hat{k})=\lambda(2\hat{i}-3\hat{j}+4\hat{k})$$

for some nonzero scalar λ.

Now, $a\hat{i}+6\hat{j}-8\hat{k}=2\lambda\hat{i}-3\lambda\hat{j}+4\lambda\hat{k}$

$\Leftrightarrow \quad 2\lambda = a, -3\lambda = 6$ and $4\lambda = -8$

$\Leftrightarrow \quad \lambda = \frac{a}{2}$ and $\lambda = -2 \Leftrightarrow \frac{a}{2} = -2 \Leftrightarrow a = -4.$

Hence, the given vectors are collinear when $a = -4$.

EXAMPLE 9 *Show that the points* $A(-2\hat{i} + 3\hat{j} + 5\hat{k})$, $B(\hat{i} + 2\hat{j} + 3\hat{k})$ *and* $C(7\hat{i} - \hat{k})$ *are collinear.* **[CBSE 2009]**

SOLUTION Clearly, we have

$$\overrightarrow{AB} = (\text{position vector of } B) - (\text{position vector of } A)$$
$$= (\hat{i} + 2\hat{j} + 3\hat{k}) - (-2\hat{i} + 3\hat{j} + 5\hat{k}) = (3\hat{i} - \hat{j} - 2\hat{k}).$$

$$\overrightarrow{BC} = (\text{position vector of } C) - (\text{position vector of } B)$$
$$= (7\hat{i} - \hat{k}) - (\hat{i} + 2\hat{j} + 3\hat{k}) = (6\hat{i} - 2\hat{j} - 4\hat{k}).$$

$\therefore \quad \overrightarrow{AB} = 2\overrightarrow{BC}$, which shows that $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are parallel vectors, having a common end point B.

Hence, the points A, B and C are collinear.

EXAMPLE 10 *Write the direction cosines of the vector* $(-2\hat{i} + \hat{j} - 5\hat{k})$. **[CBSE 2011]**

SOLUTION The given vector is $\vec{a} = (-2\hat{i} + \hat{j} - 5\hat{k})$.

Direction ratios of $\vec{a}$ are $-2, 1, -5$.

$$|\vec{a}| = \sqrt{(-2)^2 + 1^2 + (-5)^2} = \sqrt{30}.$$

Hence, the direction cosines of $\vec{a}$ are $\frac{-2}{\sqrt{30}}, \frac{1}{\sqrt{30}}, \frac{-5}{\sqrt{30}}$.

EXAMPLE 11 *What is the cosine of the angle which the vector* $(\sqrt{2}\hat{i} + \hat{j} + \hat{k})$ *makes with the y-axis?* **[CBSE 2010]**

SOLUTION The given vector is $\vec{a} = (\sqrt{2}\hat{i} + \hat{j} + \hat{k})$.

Direction ratios of $\vec{a}$ are $\sqrt{2}, 1, 1$.

$$|\vec{a}| = \sqrt{(\sqrt{2})^2 + 1^2 + 1^2} = \sqrt{4} = 2.$$

$\therefore$ direction cosines of $\vec{a}$ are $\frac{\sqrt{2}}{2}, \frac{1}{2}, \frac{1}{2}$.

Let $\vec{a}$ make angle β with the y-axis.

Then, clearly $\cos\beta = \frac{1}{2}$.

EXAMPLE 12 *Find the value of p for which* $p(\hat{i}+\hat{j}+\hat{k})$ *is a unit vector.* **[CBSE 2009]**

SOLUTION $p(\hat{i}+\hat{j}+\hat{k})$ is a unit vector

$$\Leftrightarrow \left|p(\hat{i}+\hat{j}+\hat{k})\right|^2 = 1 \Leftrightarrow (p^2+p^2+p^2) = 1$$

$$\Leftrightarrow 3p^2 = 1 \Leftrightarrow p^2 = \frac{1}{3} \Leftrightarrow p = \pm\frac{1}{\sqrt{3}}.$$

EXAMPLE 13 *If* $A(1, 2, -3)$ *and* $B(-1, -2, 1)$ *are the two given points in space then find (i) the direction ratios of* $\overrightarrow{AB}$ *and (ii) the direction cosines of* $\overrightarrow{AB}$. *Express* $\overrightarrow{AB}$ *in terms of* $\hat{i}, \hat{j}$ *and* $\hat{k}$.

SOLUTION (i) The direction ratios of $\overrightarrow{AB}$ are

$(-1-1), (-2-2), (1+3)$, i.e., –2, –4, 4.

(ii) $|\overrightarrow{AB}| = \sqrt{(-2)^2+(-4)^2+4^2} = \sqrt{36} = 6.$

$\therefore$ the direction cosines of $\overrightarrow{AB}$ are

$$\frac{-2}{6}, \frac{-4}{6}, \frac{4}{6}, \text{ i.e., } \frac{-1}{3}, \frac{-2}{3}, \frac{2}{3}.$$

Clearly, $\overrightarrow{AB} = -\frac{1}{3}\hat{i} - \frac{2}{3}\hat{j} + \frac{2}{3}\hat{k}.$

EXAMPLE 14 *Find the scalar and vector components of the vector with initial point* $A(3, -1, 2)$ *and terminal point* $B(-5, 4, 3)$.

SOLUTION We have

$$\overrightarrow{AB} = (\textit{position vector of } B) - (\textit{position vector of } A)$$
$$= (-5\hat{i}+4\hat{j}+3\hat{k}) - (3\hat{i}-\hat{j}+2\hat{k}) = (-8\hat{i}+5\hat{j}+\hat{k}).$$

The scalar components of $\overrightarrow{AB}$ are –8, 5, 1.

The vector components of $\overrightarrow{AB}$ are $-8\hat{i}, 5\hat{j}, \hat{k}$.

EXAMPLE 15 *Write two different vectors having same magnitude.*

SOLUTION Consider the vectors $\vec{a} = 2\hat{i}+3\hat{j}-4\hat{k}$ and $\vec{b} = 4\hat{i}+2\hat{j}+3\hat{k}.$

Clearly, $\vec{a} \neq \vec{b}$.

But, $|\vec{a}| = \sqrt{2^2+3^2+(-4)^2} = \sqrt{29}$ and $|\vec{b}| = \sqrt{4^2+2^2+3^2} = \sqrt{29}.$

Thus, $|\vec{a}| = |\vec{b}|$ and $\vec{a} \neq \vec{b}$.

EXAMPLE 16 *Write two different vectors having same direction.*

SOLUTION Clearly, $3\overrightarrow{AB}$ and $5\overrightarrow{AB}$ are two different vectors having the same direction.

EXAMPLE 17 *Show that the points with position vectors* $\vec{a} = (3\hat{i} - 4\hat{j} - 4\hat{k})$, $\vec{b} = (2\hat{i} - \hat{j} + \hat{k})$ *and* $\vec{c} = (\hat{i} - 3\hat{j} - 5\hat{k})$ *respectively, form the vertices of a right-angled triangle.*

SOLUTION We have

$$\begin{aligned}\overrightarrow{AB} &= (\text{position vector of } B) - (\text{position vector of } A)\\ &= (\vec{b} - \vec{a}) = (2\hat{i} - \hat{j} + \hat{k}) - (3\hat{i} - 4\hat{j} - 4\hat{k})\\ &= (-\hat{i} + 3\hat{j} + 5\hat{k}).\end{aligned}$$

$$\begin{aligned}\overrightarrow{BC} &= (\text{position vector of } C) - (\text{position vector of } B)\\ &= (\vec{c} - \vec{b}) = (\hat{i} - 3\hat{j} - 5\hat{k}) - (2\hat{i} - \hat{j} + \hat{k})\\ &= (-\hat{i} - 2\hat{j} - 6\hat{k}).\end{aligned}$$

$$\begin{aligned}\overrightarrow{CA} &= (\text{position vector of } A) - (\text{position vector of } C)\\ &= (\vec{a} - \vec{c}) = (3\hat{i} - 4\hat{j} - 4\hat{k}) - (\hat{i} - 3\hat{j} - 5\hat{k})\\ &= (2\hat{i} - \hat{j} + \hat{k}).\end{aligned}$$

$\therefore \quad |\overrightarrow{AB}|^2 = \{(-1)^2 + 3^2 + 5^2\} = (1 + 9 + 25) = 35,$

$|\overrightarrow{BC}|^2 = \{(-1)^2 + (-2)^2 + (-6)^2\} = (1 + 4 + 36) = 41,$

$|\overrightarrow{CA}|^2 = \{2^2 + (-1)^2 + 1^2\} = (4 + 1 + 1) = 6.$

$\therefore \quad |AB|^2 + |CA|^2 = |BC|^2 \Rightarrow AB^2 + CA^2 = BC^2.$

Hence, $\triangle ABC$ is right-angled at A.

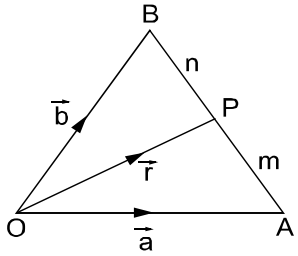
Section Formulae

THEOREM 1 (Section Formula for Internal Division) *Let A and B be two points with position vectors $\vec{a}$ and $\vec{b}$ respectively and let P be a point dividing AB internally in the ratio $m : n$. Let $\overrightarrow{OP} = \vec{r}$. Then, prove that*

$$\vec{r} = \frac{(m\vec{b} + n\vec{a})}{(m+n)}.$$

PROOF Let O be the origin. Then, $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{OB} = \vec{b}$.

Let P be a point on AB such that $\frac{AP}{PB} = \frac{m}{n}$. Then,



$$\frac{AP}{PB} = \frac{m}{n}$$

$$\Rightarrow \; n \cdot AP = m \cdot PB$$

$$\Rightarrow \; n(\overrightarrow{AP}) = m(\overrightarrow{PB})$$

$$\Rightarrow \; n(\overrightarrow{OP} - \overrightarrow{OA}) = m(\overrightarrow{OB} - \overrightarrow{OP})$$

$$\Rightarrow \; (m+n)\,\overrightarrow{OP} = m\,\overrightarrow{OB} + n\,\overrightarrow{OA} = m\,\vec{b} + n\,\vec{a}$$

$$\Rightarrow \; \overrightarrow{OP} = \frac{m\,\vec{b} + n\,\vec{a}}{(m+n)} \quad \Rightarrow \quad \vec{r} = \frac{(m\,\vec{b} + n\,\vec{a})}{(m+n)}.$$

COROLLARY *The position vector of the midpoint of the join of two points with position vectors* $\vec{a}$ *and* $\vec{b}$ *is* $\frac{1}{2}(\vec{a} + \vec{b})$.

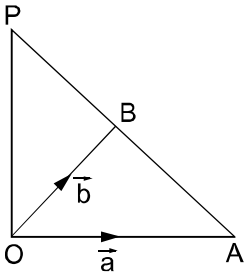
PROOF Let A and B be two points with position vectors $\vec{a}$ and $\vec{b}$ respectively.
Let P be the midpoint of AB.
Then, P divides AB in the ratio $1 : 1$.

$$\therefore \; \overrightarrow{OP} = \frac{(1 \cdot \vec{b} + 1 \cdot \vec{a})}{(1+1)} = \frac{1}{2}(\vec{a} + \vec{b}).$$

THEOREM 2 (Section Formula for External Division) *Let A and B be two points with position vectors* $\vec{a}$ *and* $\vec{b}$ *respectively and let P be a point dividing AB externally in the ratio $m : n$. Let* $\overrightarrow{OP} = \vec{r}$. *Then, prove that*

$$\vec{r} = \frac{(m\,\vec{b} - n\,\vec{a})}{(m-n)}.$$

PROOF Let O be the origin. Then, $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{OB} = \vec{b}$.
Let AB be produced to P such that $AP : BP = m : n$.
Now, $\frac{AP}{BP} = \frac{m}{n}$



$$\Rightarrow \quad n \cdot AP = m \cdot BP \quad \Rightarrow \quad n \cdot \overrightarrow{AP} = m \cdot \overrightarrow{BP}$$

$$\Rightarrow \quad n(\overrightarrow{OP} - \overrightarrow{OA}) = m(\overrightarrow{OP} - \overrightarrow{OB})$$

$$\Rightarrow \quad (m-n)\,\overrightarrow{OP} = (m\,\overrightarrow{OB} - n\,\overrightarrow{OA}) = (m\,\vec{b} - n\,\vec{a})$$

$$\Rightarrow \quad \overrightarrow{OP} = \frac{(m\,\vec{b} - n\,\vec{a})}{(m-n)} \quad \Rightarrow \quad \vec{r} = \frac{(m\,\vec{b} - n\,\vec{a})}{(m-n)}.$$

SOLVED EXAMPLES

EXAMPLE 1 *Find the position vector of a point R which divides the line joining the points* $P(\hat{i}+2\hat{j}-\hat{k})$ *and* $Q(-\hat{i}+\hat{j}+\hat{k})$ *in the ratio* $2:1$*, (i) internally and (ii) externally.*

SOLUTION Here $\vec{a}=(\hat{i}+2\hat{j}-\hat{k})$ and $\vec{b}=(-\hat{i}+\hat{j}+\hat{k})$. Also, $m=2, n=1$.

(i) When R divides PQ internally in the ratio $2:1$; then

$$\text{position vector of } R=\frac{(m\vec{b}+n\vec{a})}{(m+n)}$$

$$=\frac{2(-\hat{i}+\hat{j}+\hat{k})+1\cdot(\hat{i}+2\hat{j}-\hat{k})}{(2+1)}$$

$$=\frac{(-\hat{i}+4\hat{j}+\hat{k})}{3}.$$

(ii) When R divides PQ externally in the ratio $2:1$; then

$$\text{position vector of } R=\frac{(m\vec{b}-n\vec{a})}{(m-n)}$$

$$=\frac{2(-\hat{i}+\hat{j}+\hat{k})+1\cdot(\hat{i}+2\hat{j}-\hat{k})}{(2-1)}$$

$$=(-3\hat{i}+3\hat{k})\cdot$$

EXAMPLE 2 *P and Q are two points with position vectors* $(3\vec{a}-2\vec{b})$ *and* $(\vec{a}+\vec{b})$ *respectively. Write the position vector of a point R which divides the line segment PQ in the ratio $2:1$ externally.* **[CBSE 2013]**

SOLUTION The position vectors of the given points are

$$P(3\vec{a}-2\vec{b}) \text{ and } Q(\vec{a}+\vec{b}).$$

We have to divide PQ in the ratio $2:1$ externally at the point R.

The position vector of R is

$$\frac{2(\vec{a}+\vec{b})-1\cdot(3\vec{a}-2\vec{b})}{(2-1)}=(-\vec{a}+4\vec{b}).$$

Hence, the position vector of R is $(-\vec{a}+4\vec{b})$.

EXAMPLE 3 *Find the position vector of a point R which divides the line segment joining the points* $A(2,-3,4)$ *and* $B(3,1,-2)$ *externally in the ratio* $3:2$.

SOLUTION The position vector of A is $(2\hat{i} - 3\hat{j} + 4\hat{k})$.

The position vector of B is $(3\hat{i} + \hat{j} - 2\hat{k})$.

Let R divide AB externally in the ratio $3 : 2$.

Then, position vector of R

$$= \left(\frac{3\vec{b} - 2\vec{a}}{3-2}\right) = \frac{3(3\hat{i} + \hat{j} - 2\hat{k}) - 2(2\hat{i} - 3\hat{j} + 4\hat{k})}{1}$$

$$= (5\hat{i} + 9\hat{j} - 14\hat{k}).$$

Hence, the position vector of R is $(5\hat{i} + 9\hat{j} - 14\hat{k})$.

EXAMPLE 4 *Find the position vector of the mid-point of the vector joining the points* $P(2, 3, 4)$ *and* $Q(4, 1, -2)$. **[CBSE 2011]**

SOLUTION The position vectors of the given points P and Q are

$\vec{a} = (2\hat{i} + 3\hat{j} + 4\hat{k})$ and $\vec{b} = (4\hat{i} + \hat{j} - 2\hat{k})$ respectively.

EXAMPLE 5 *Show that the three points* $A(1, -2, -8)$, $B(5, 0, -2)$ *and* $C(11, 3, 7)$ *are collinear and find the ratio in which B divides AC.*

SOLUTION The position vectors of A, B and C are $(\hat{i} - 2\hat{j} - 8\hat{k})$, $(5\hat{i} - 2\hat{k})$ and $(11\hat{i} + 3\hat{j} + 7\hat{k})$ respectively.

$\therefore$ $\overrightarrow{AB} = (\text{position vector of } B) - (\text{position vector of } A)$

$$= (5\hat{i} - 2\hat{k}) - (\hat{i} - 2\hat{j} + 8\hat{k}) = (4\hat{i} + 2\hat{j} + 6\hat{k}).$$

$\overrightarrow{BC} = (\text{position vector of } C) - (\text{position vector of } B)$

$$= (11\hat{i} + 3\hat{j} + 7\hat{k}) - (5\hat{i} - 2\hat{k}) = (6\hat{i} + 3\hat{j} + 9\hat{k}), \text{ and}$$

$\overrightarrow{AC} = (\text{position vector of } C) - (\text{position vector of } A)$

$$= (11\hat{i} + 3\hat{j} + 7\hat{k}) - (\hat{i} - 2\hat{j} - 8\hat{k}) = (10\hat{i} + 5\hat{j} + 15\hat{k}).$$

Now, $\overrightarrow{AB} = (4\hat{i} + 2\hat{j} + 6\hat{k}) = 2(2\hat{i} + \hat{j} + 3\hat{k})$

$$= \frac{2}{5}(10\hat{i} + 5\hat{j} + 15\hat{k}) = \frac{2}{5}\overrightarrow{AC}.$$

$\therefore$ $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are parallel vectors having same end point A.

Hence, the points A, B and C are collinear.

Also, $\overrightarrow{AB} = (4\hat{i} + 2\hat{j} + 6\hat{k}) = 2(2\hat{i} + \hat{j} + 3\hat{k})$

$$= \frac{2}{3}(6\hat{i} + 3\hat{j} + 9\hat{k}) = \frac{2}{3}\overrightarrow{BC}.$$

$$\therefore \quad \frac{|\overrightarrow{AB}|}{|\overrightarrow{BC}|} = \frac{2}{3}.$$

Hence, B divides AC in the ratio 2 : 3.

EXERCISE 22

1. Write down the magnitude of each of the following vectors:

   (i) $\vec{a} = \hat{i} + 2\hat{j} + 5\hat{k}$ (ii) $\vec{b} = 5\hat{i} - 4\hat{j} - 3\hat{k}$

   (iii) $\vec{c} = \left(\frac{1}{\sqrt{3}}\hat{i} - \frac{1}{\sqrt{3}}\hat{j} + \frac{1}{\sqrt{3}}\hat{k}\right)$ (iv) $\vec{d} = (\sqrt{2}\hat{i} + \sqrt{3}\hat{j} - \sqrt{5}\hat{k})$

2. Find a unit vector in the direction of the vector:

   (i) $(3\hat{i} + 4\hat{j} - 5\hat{k})$ (ii) $(3\hat{i} - 2\hat{j} + 6\hat{k})$ **[CBSE 2012]**

   (iii) $(\hat{i} + \hat{k})$ (iv) $(2\hat{i} + \hat{j} + 2\hat{k})$ **[CBSE 2009]**

3. If $\vec{a} = (2\hat{i} - 4\hat{j} + 5\hat{k})$ then find the value of λ so that $\lambda\vec{a}$ may be a unit vector.

4. If $\vec{a} = (-\hat{i} + \hat{j} - \hat{k})$ and $\vec{b} = (2\hat{i} - \hat{j} + 2\hat{k})$ then find the unit vector in the direction of $(\vec{a} + \vec{b})$.

5. If $\vec{a} = (3\hat{i} + \hat{j} - 5\hat{k})$ and $\vec{b} = (\hat{i} + 2\hat{j} - \hat{k})$ then find a unit vector in the direction of $(\vec{a} - \vec{b})$.

6. If $\vec{a} = (\hat{i} + 2\hat{j} - 3\hat{k})$ and $\vec{b} = (2\hat{i} + 4\hat{j} + 9\hat{k})$ then find a unit vector parallel to $(\vec{a} + \vec{b})$. **[CBSE 2008]**

7. Find a vector of magnitude 9 units in the direction of the vector $(-2\hat{i} + \hat{j} + 2\hat{k})$. **[CBSE 2010]**

8. Find a vector of magnitude 8 units in the direction of the vector $(5\hat{i} - \hat{j} + 2\hat{k})$.

9. Find a vector of magnitude 21 units in the direction of the vector $(2\hat{i} - 3\hat{j} + 6\hat{k})$. **[CBSE 2014]**

10. If $\vec{a} = (\hat{i} - 2\hat{j})$, $\vec{b} = (2\hat{i} - 3\hat{j})$ and $\vec{c} = (2\hat{i} + 3\hat{k})$, find $(\vec{a} + \vec{b} + \vec{c})$.

[CBSE 2012]

11. If $A(-2, 1, 2)$ and $B(2, -1, 6)$ are two given points, find a unit vector in the direction of $\overrightarrow{AB}$.

12. Find the direction ratios and direction cosines of the vector $\vec{a} = (5\hat{i} - 3\hat{j} + 4\hat{k})$.

13. Find the direction ratios and the direction cosines of the vector joining the points $A(2, 1, -2)$ and $B(3, 5, -4)$.

14. Show that the points A, B and C having position vectors $(\hat{i} + 2\hat{j} + 7\hat{k})$, $(2\hat{i} + 6\hat{j} + 3\hat{k})$ and $(3\hat{i} + 10\hat{j} - 3\hat{k})$ respectively, are collinear.

15. The position vectors of the points A, B and C are $(2\hat{i} + \hat{j} - \hat{k})$, $(3\hat{i} - 2\hat{j} + \hat{k})$ and $(\hat{i} + 4\hat{j} - 3\hat{k})$ respectively. Show that the points A, B and C are collinear.

16. If the position vectors of the vertices A, B and C of a ΔABC be $(\hat{i} + 2\hat{j} + 3\hat{k})$, $(2\hat{i} + 3\hat{j} + \hat{k})$ and $(3\hat{i} + \hat{j} + 2\hat{k})$ respectively, prove that ΔABC is equilateral.

17. Show that the points A, B and C having position vectors $(3\hat{i} - 4\hat{j} - 4\hat{k})$, $(2\hat{i} - \hat{j} + \hat{k})$ and $(\hat{i} - 3\hat{j} - 5\hat{k})$ respectively, form the vertices of a right-angled triangle.

18. Using vector method, show that the points $A(1, -1, 0)$, $B(4, -3, 1)$ and $C(2, -4, 5)$ are the vertices of a right-angled triangle.

19. Find the position vector of the point which divides the join of the points $(2\vec{a} - 3\vec{b})$ and $(3\vec{a} - 2\vec{b})$ (i) internally and (ii) externally in the ratio 2 : 3.

20. The position vectors of two points A and B are $(2\vec{a} + \vec{b})$ and $(\vec{a} - 3\vec{b})$ respectively. Find the position vector of a point C which divides AB externally in the ratio 1 : 2. Also, show that A is the mid-point of the line segment CB. **[CBSE 2010]**

21. Find the position vector of a point R which divides the line joining $A(-2, 1, 3)$ and $B(3, 5, -2)$ in the ratio 2 : 1 (i) internally (ii) externally.

22. Find the position vector of the mid-point of the vector joining the points

$$A(3\hat{i} + 2\hat{j} + 6\hat{k}) \text{ and } B(\hat{i} + 4\hat{j} - 2\hat{k}).$$

23. If $\overrightarrow{AB} = (2\hat{i} + \hat{j} - 3\hat{k})$ and $A(1, 2, -1)$ is the given point, find the coordinates of B.

24. Write a unit vector in the direction of $\overrightarrow{PQ}$, where P and Q are the points $(1, 3, 0)$ and $(4, 5, 6)$ respectively. **[CBSE 2014]**

ANSWERS (EXERCISE 22)

1. (i) $\sqrt{30}$ (ii) $5\sqrt{2}$ (iii) 1 (iv) $\sqrt{10}$

2. (i) $\left(\frac{3}{5\sqrt{2}}\hat{i}+\frac{4}{5\sqrt{2}}\hat{j}-\frac{5}{5\sqrt{2}}\hat{k}\right)$ (ii) $\left(\frac{3}{7}\hat{i}-\frac{2}{7}\hat{j}+\frac{6}{7}\hat{k}\right)$

(iii) $\left(\frac{1}{\sqrt{2}}\hat{i}+\frac{1}{\sqrt{2}}\hat{k}\right)$ (iv) $\left(\frac{2}{3}\hat{i}+\frac{1}{3}\hat{j}+\frac{2}{3}\hat{k}\right)$

3. $\pm\frac{1}{3\sqrt{5}}$ **4.** $\frac{1}{\sqrt{2}}(\hat{i}+\hat{k})$

5. $\frac{1}{\sqrt{21}}(2\hat{i}-\hat{j}-4\hat{k})$ **6.** $\pm\frac{1}{3}(\hat{i}+2\hat{j}+2\hat{k})$

7. $(-6\hat{i}+3\hat{j}+6\hat{k})$ **8.** $\frac{8}{\sqrt{30}}(5\hat{i}-\hat{j}+2\hat{k})$

9. $(6\hat{i}-9\hat{j}+18\hat{k})$ **10.** $(5\hat{i}-5\hat{j}+3\hat{k})$

11. $\left(\frac{2}{3}\hat{i}-\frac{1}{3}\hat{j}+\frac{2}{3}\hat{k}\right)$ **12.** $(5,-3,4),\left(\frac{1}{\sqrt{2}},\frac{-3}{5\sqrt{2}},\frac{4}{5\sqrt{2}}\right)$

13. $(1,4,-2),\left(\frac{1}{\sqrt{21}},\frac{4}{\sqrt{21}},\frac{-2}{\sqrt{21}}\right)$ **19.** (i) $\frac{12}{5}\vec{a}-\frac{13}{5}\vec{b}$ (ii) $-5\vec{b}$

20. $(3\vec{a}+5\vec{b})$ **21.** (i) $\left(\frac{4}{3}\hat{i}+\frac{11}{3}\hat{j}-\frac{1}{3}\hat{k}\right)$ (ii) $(8\hat{i}+9\hat{j}-7\hat{k})$

22. $P(2\hat{i}+3\hat{j}+2\hat{k})$ **23.** $(3,3,-4)$ **24.** $\frac{1}{7}(3\hat{i}+2\hat{j}+6\hat{k})$

HINTS TO SOME SELECTED QUESTIONS

16. Show that $|\overrightarrow{AB}|=|\overrightarrow{BC}|=|\overrightarrow{CA}|$.

17. Show that $\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CA}=\vec{0}$ and $|\overrightarrow{AB}|^2+|\overrightarrow{CA}|^2=|\overrightarrow{BC}|^2$.

$\therefore$ $\triangle ABC$ is right angled at A.

18. The position vectors of A, B, C are $(\hat{i}-\hat{j})$, $(4\hat{i}-3\hat{j}+\hat{k})$ and $(2\hat{i}-4\hat{j}+5\hat{k})$ respectively. Show that $\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CA}=\vec{0}$ and $|\overrightarrow{AB}|^2+|\overrightarrow{BC}|^2=|\overrightarrow{CA}|^2$.

$\therefore$ $\triangle ABC$ is right angled at B.

23. Let O be the origin. Then, $\overrightarrow{OA}=(\hat{i}+2\hat{j}-\hat{k})$.

$\therefore$ $\overrightarrow{AB}=(\overrightarrow{OB}-\overrightarrow{OA})\Rightarrow\overrightarrow{OB}=(\overrightarrow{AB}+\overrightarrow{OA})$.

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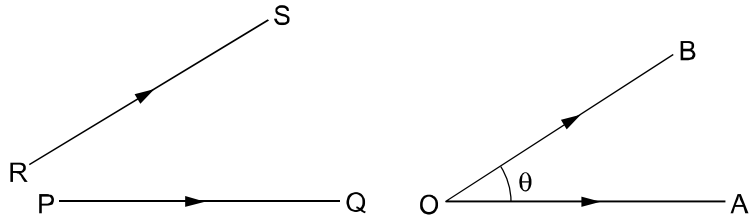
23. SCALAR, OR DOT, PRODUCT OF VECTORS

ANGLE BETWEEN TWO VECTORS Let $\overrightarrow{PQ}$ and $\overrightarrow{RS}$ be two given vectors. Take any point O, and draw $OA \,||\, PQ$ and $OB \,||\, RS$. Then, $\angle AOB = \theta$ is called the *angle between* $\overrightarrow{PQ}$ *and* $\overrightarrow{RS}$, provided $0 \le \theta \le \pi$.



If $\theta = 0$ or $\theta = \pi$ then $\overrightarrow{PQ} \,||\, \overrightarrow{RS}$.

If $\theta = \dfrac{\pi}{2}$ then $\overrightarrow{PQ}$ and $\overrightarrow{RS}$ are called *perpendicular*, or *orthogonal*, *vectors*.

SCALAR PRODUCT, OR DOT PRODUCT, OF TWO VECTORS Let $\vec{a}$ and $\vec{b}$ be two vectors, and let θ be the angle between them. Then, the *scalar product*, or *dot product*, of $\vec{a}$ and $\vec{b}$, denoted by $\vec{a} \cdot \vec{b}$, is defined as $\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}| \cos\theta = a\,b\cos\theta$.

Clearly, the scalar product of two vectors is a scalar.

ANGLE BETWEEN TWO VECTORS IN TERMS OF SCALAR PRODUCT Let θ be the angle between two nonzero vectors $\vec{a}$ and $\vec{b}$. Then,

$$\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}| \cos\theta.$$

$$\therefore \quad \cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|\,|\vec{b}|} \quad \Rightarrow \quad \theta = \cos^{-1}\left(\frac{\vec{a} \cdot \vec{b}}{|\vec{a}|\,|\vec{b}|}\right).$$

LENGTH OF A VECTOR Let $\vec{a}$ be any vector.

Then, $\vec{a} \cdot \vec{a} = |\vec{a}|\,|\vec{a}| \cos 0^\circ = |\vec{a}|^2$.

$$\therefore \quad |\vec{a}| = \sqrt{\vec{a} \cdot \vec{a}}.$$

REMARKS (i) If $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$, we define $\vec{a} \cdot \vec{b} = 0$.

(ii) If $\vec{a}$ and $\vec{b}$ are like vectors, we have $\theta = 0$.

$\therefore \quad \vec{a} \cdot \vec{b} = a\,b\cos 0^\circ = ab.$

(iii) If $\vec{a}$ and $\vec{b}$ are unlike vectors, we have $\theta = \pi$.

$\therefore \quad \vec{a} \cdot \vec{b} = a\,b\cos\pi = -ab.$

(iv) If $\vec{a}$ and $\vec{b}$ are orthogonal vectors, we have $\theta = \frac{\pi}{2}$.

$\therefore \quad \vec{a} \cdot \vec{b} = a\,b\cos\frac{\pi}{2} = 0.$

SOLVED PROBLEMS

EXAMPLE 1 *Let $\vec{a}$ and $\vec{b}$ be two given vectors such that $|\vec{a}| = 3$, $|\vec{b}| = 4$ and the angle between them is 60°. Find $\vec{a} \cdot \vec{b}$.*

SOLUTION Clearly, we have

$$\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}|\cos 60^\circ = \left(3 \times 4 \times \frac{1}{2}\right) = 6.$$

EXAMPLE 2 *Let $\vec{a}$ and $\vec{b}$ be two given vectors such that $|\vec{a}| = \sqrt{3}$, $|\vec{b}| = 2$ and $\vec{a} \cdot \vec{b} = \sqrt{6}$. Find the angle between $\vec{a}$ and $\vec{b}$.*

SOLUTION Let θ be the angle between $\vec{a}$ and $\vec{b}$. Then,

$$\vec{a} \cdot \vec{b} = \sqrt{6} \;\Rightarrow\; |\vec{a}|\,|\vec{b}|\cos\theta = \sqrt{6} \;\Rightarrow\; (\sqrt{3})(2)\cos\theta = \sqrt{6}$$

$$\Rightarrow\; \cos\theta = \frac{\sqrt{6}}{2\sqrt{3}} = \frac{1}{\sqrt{2}} \;\Rightarrow\; \theta = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}.$$

Hence, the required angle is $\frac{\pi}{4}$.

EXAMPLE 3 *If $\vec{a}$ and $\vec{b}$ are two vectors such that $|\vec{a}| = |\vec{b}| = \sqrt{2}$ and $\vec{a} \cdot \vec{b} = -1$, find the angle between $\vec{a}$ and $\vec{b}$.*

SOLUTION Let θ be the angle between $\vec{a}$ and $\vec{b}$. Then, $\vec{a} \cdot \vec{b} = -1$

$$\Leftrightarrow \quad |\vec{a}| \cdot |\vec{b}| \cdot \cos\theta = -1 \;\Leftrightarrow\; \sqrt{2} \times \sqrt{2} \times \cos\theta = -1$$

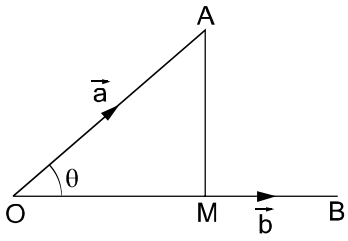
$$\Leftrightarrow \quad \cos\theta = \frac{-1}{2} \;\Leftrightarrow\; \theta = \frac{2\pi}{3} \qquad [\because \quad 0 \le \theta \le 2\pi].$$

Hence, the angle between $\vec{a}$ and $\vec{b}$ is $\frac{2\pi}{3}$.

PROJECTION

Let $\overrightarrow{OA} = \vec{a}, \overrightarrow{OB} = \vec{b}$ and $\angle BOA = \theta$. Draw $AM \perp OB$.

Then, OM is the projection of $\vec{a}$ on $\vec{b}$.



$$OM = (OA)\cos\theta = |\vec{a}|\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}.$$

$$\therefore \quad \text{projection of } \vec{a} \text{ on } \vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}.$$

PROPERTIES OF SCALAR PRODUCT

THEOREM 1 (Commutative Law) *Prove that* $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$.

PROOF If $\vec{a}$ or $\vec{b}$ is a zero vector then $\vec{a}\cdot\vec{b} = 0$ and $\vec{b}\cdot\vec{a} = 0$.

So, in this case, $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$.

Now, let $\vec{a}$ and $\vec{b}$ be any two nonzero vectors, and let θ be the angle between them. Then,

$$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta = a\,b\cos\theta, \quad \text{and}$$

$$\vec{b}\cdot\vec{a} = |\vec{b}||\vec{a}|\cos(-\theta) = b\,a\cos\theta = a\,b\cos\theta.$$

So, in this case also, $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$.

Hence, $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$.

THEOREM 2 *Prove that* $\vec{a}\cdot\vec{b} = 0 \Leftrightarrow \vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$ or $\vec{a} \perp \vec{b}$.

PROOF Let $\vec{a}\cdot\vec{b} = 0$.

Then, $\vec{a}\cdot\vec{b} = 0 \Rightarrow a\,b\cos\theta = 0 \Rightarrow a = 0$ or $b = 0$ or $\cos\theta = 0$

$$\Rightarrow a = 0 \text{ or } b = 0 \text{ or } \theta = \frac{\pi}{2}$$

$$\Rightarrow \vec{a} = \vec{0} \text{ or } \vec{b} = \vec{0} \text{ or } \vec{a} \perp \vec{b}.$$

Thus, $\vec{a}\cdot\vec{b} = 0 \Rightarrow \vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$ or $\vec{a} \perp \vec{b}$.

Conversely, let $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$ or $\vec{a} \perp \vec{b}$.

If $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$ then by definition, $\vec{a}\cdot\vec{b} = 0$.

If $\vec{a} \perp \vec{b}$ then $\vec{a} \cdot \vec{b} = a\,b\cos\frac{\pi}{2} = 0.$

$\therefore \quad \vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$ or $\vec{a} \perp \vec{b} \Rightarrow \vec{a} \cdot \vec{b} = 0.$

Hence, $\vec{a} \cdot \vec{b} = 0 \Leftrightarrow \vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$ or $\vec{a} \perp \vec{b}.$

THEOREM 3 *For any two vectors $\vec{a}$ and $\vec{b}$, prove that:*

(i) $\vec{a} \cdot (-\vec{b}) = -(\vec{a} \cdot \vec{b}) = (-\vec{a}) \cdot \vec{b}$

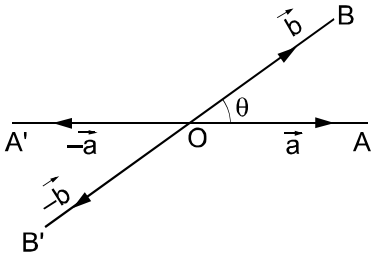
(ii) $(-\vec{a}) \cdot (-\vec{b}) = \vec{a} \cdot \vec{b}$

PROOF Let $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{OB} = \vec{b}$. Produce AO and BO to A' and B' respectively, such that $OA' = OA$ and $OB' = OB.$

Then, $\overrightarrow{OA'} = -\vec{a}$ and $\overrightarrow{OB'} = -\vec{b}.$

Let $\angle AOB = \theta.$

Then, $\angle AOB' = \pi - \theta.$



(i) $\vec{a} \cdot (-\vec{b}) = |\vec{a}|\,|-\vec{b}|\cos(\pi - \theta)$

$= |\vec{a}|\,|\vec{b}| \cdot (-\cos\theta)$

$= -a\,b\cos\theta = -(\vec{a} \cdot \vec{b}).$

$\therefore \quad \vec{a} \cdot (-\vec{b}) = -(\vec{a} \cdot \vec{b}).$

Similarly, $(-\vec{a}) \cdot \vec{b} = -(\vec{a} \cdot \vec{b}).$

Hence, $\vec{a} \cdot (-\vec{b}) = (-\vec{a}) \cdot \vec{b} = -(\vec{a} \cdot \vec{b}).$

(ii) $(-\vec{a}) \cdot (-\vec{b}) = |-\vec{a}|\,|-\vec{b}|\cos\angle A'OB'$

$= |\vec{a}|\,|\vec{b}|\cos\theta \quad [\because \angle A'OB' = \theta]$

$= a\,b\cos\theta = \vec{a} \cdot \vec{b}.$

Hence, $(-\vec{a}) \cdot (-\vec{b}) = (\vec{a} \cdot \vec{b}).$

THEOREM 4 (Distributive Law) *Prove that* $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}.$

PROOF Let O be the origin, $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$ and $\overrightarrow{BC} = \vec{c}$. Then,

$\overrightarrow{OC} = (\overrightarrow{OB} + \overrightarrow{BC}) = (\vec{b} + \vec{c}).$

Draw $BM \perp OA$ and $CN \perp OA$. Then,

$\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \overrightarrow{OC} = |\vec{a}|\{(OC)\cos\theta\}$, where $\angle AOC = \theta$

$= a \times ON \quad [\because (OC)\cos\theta = ON]$

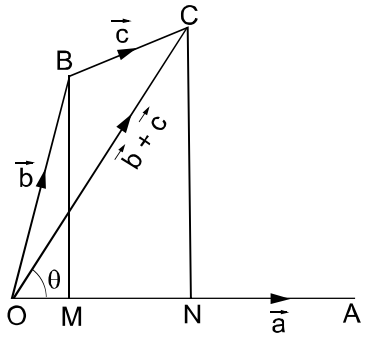
$= a(OM + MN)$

$= a(OM) + a(MN)$

$= a(\text{projection of } \vec{b} \text{ on } \vec{a})$

$\quad + a(\text{projection of } \vec{c} \text{ on } \vec{a})$

$= (\vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c})$.



Hence, $\vec{a} \cdot (\vec{b} + \vec{c}) = (\vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c})$.

THEOREM 5 (Cauch Schwartz Inequality) *Prove that* $|\vec{a} \cdot \vec{b}| \leq |\vec{a}||\vec{b}|$.

PROOF When $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$, then $|\vec{a} \cdot \vec{b}| = 0 = |\vec{a}||\vec{b}|$.

So, let us assume that $\vec{a} \neq \vec{0}$ and $\vec{b} \neq \vec{0}$. Then,

$$(\vec{a} \cdot \vec{b}) = |\vec{a}||\vec{b}| \cos\theta \Rightarrow |\vec{a} \cdot \vec{b}| = |\vec{a}||\vec{b}||\cos\theta|$$

$$\Rightarrow \frac{|\vec{a} \cdot \vec{b}|}{|\vec{a}||\vec{b}|} = |\cos\theta| \leq 1 \Rightarrow |\vec{a} \cdot \vec{b}| \leq |\vec{a}||\vec{b}|.$$

Hence, $|\vec{a} \cdot \vec{b}| \leq |\vec{a}||\vec{b}|$.

THEOREM 6 (Triangle Inequality) *Prove that* $|\vec{a} + \vec{b}| \leq |\vec{a}| + |\vec{b}|$.

PROOF When $\vec{a} = \vec{0}$, then $|\vec{a}| = 0$.

$\therefore |\vec{a} + \vec{b}| = |\vec{0} + \vec{b}| = |\vec{b}|$ and $|\vec{a}| + |\vec{b}| = 0 + |\vec{b}| = |\vec{b}|$.

So, in this case, $|\vec{a} + \vec{b}| = |\vec{a}| + |\vec{b}|$.

Similarly, when $\vec{b} = \vec{0}$, we have $|\vec{a} + \vec{b}| = |\vec{a}| + |\vec{b}|$.

Let us consider the case when $\vec{a} \neq \vec{0}$ and $\vec{b} \neq \vec{0}$.

Now, $|\vec{a} + \vec{b}|^2 = (\vec{a} + \vec{b})^2$

$= (\vec{a} + \vec{b}) \cdot (\vec{a} + \vec{b})$

$= \vec{a} \cdot \vec{a} + \vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{a} + \vec{b} \cdot \vec{b}$

$= |\vec{a}|^2 + 2(\vec{a} \cdot \vec{b}) + |\vec{b}|^2 \quad [\because \vec{b} \cdot \vec{a} = \vec{a} \cdot \vec{b}]$

$\leq |\vec{a}|^2 + 2|\vec{a} \cdot \vec{b}| + |\vec{b}|^2 \quad [\because \alpha \leq |\alpha| \forall \alpha \in R]$

$\leq |\vec{a}|^2 + 2|\vec{a}| \cdot |\vec{b}| + |\vec{b}|^2 = \{|\vec{a}| + |\vec{b}|\}^2$.

$[\because |\vec{a} \cdot \vec{b}| \leq |\vec{a}| \cdot |\vec{b}|]$

$\therefore |\vec{a} + \vec{b}| \leq |\vec{a}| + |\vec{b}|$.

ORTHONORMAL VECTOR TRIAD

Let $\hat{i}, \hat{j}, \hat{k}$ be unit vectors along three mutually perpendicular coordinate axes namely, the x-axis, y-axis and z-axis respectively. Then, these vectors are said to form an orthonormal triad of vectors.

Clearly, we have:

(i) $\hat{i} \cdot \hat{i} = |\hat{i}||\hat{i}| \cos 0^\circ = 1.$

Similarly, $\hat{j} \cdot \hat{j} = 1$ and $\hat{k} \cdot \hat{k} = 1.$

Thus, $\hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1.$

(ii) $\hat{i} \cdot \hat{j} = |\hat{i}||\hat{j}| \cos \frac{\pi}{2} = 0.$

Similarly, $\hat{i} \cdot \hat{k} = 0,\ \hat{j} \cdot \hat{k} = 0,\ \hat{j} \cdot \hat{i} = 0,\ \hat{k} \cdot \hat{i} = 0$ and $\hat{k} \cdot \hat{j} = 0.$

Hence, $\hat{i} \cdot \hat{j} = \hat{i} \cdot \hat{k} = \hat{j} \cdot \hat{i} = \hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{i} = \hat{k} \cdot \hat{j} = 0.$

SUMMARY

$\hat{i}, \hat{j}, \hat{k}$ are unit vectors such that:

(i) $\hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1.$

(ii) $\hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{i} = 0,\ \hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{j} = 0$ and $\hat{i} \cdot \hat{k} = \hat{k} \cdot \hat{i} = 0.$

THEOREM 7 *If* $\vec{a} = a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}$ *and* $\vec{b} = b_1 \hat{i} + b_2 \hat{j} + b_3 \hat{k}$ *the prove that* $\vec{a} \cdot \vec{b} = (a_1 b_1 + a_2 b_2 + a_3 b_3)$.

PROOF We have

$$\vec{a} \cdot \vec{b} = (a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}) \cdot (b_1 \hat{i} + b_2 \hat{j} + b_3 \hat{k})$$

$$= a_1 \hat{i} \cdot (b_1 \hat{i} + b_2 \hat{j} + b_3 \hat{k}) + a_2 \hat{j} \cdot (b_1 \hat{i} + b_2 \hat{j} + b_3 \hat{k}) + a_3 \hat{k} \cdot (b_1 \hat{i} + b_2 \hat{j} + b_3 \hat{k})$$

$$= (a_1 b_1)(\hat{i} \cdot \hat{i}) + (a_1 b_2)(\hat{i} \cdot \hat{j}) + (a_1 b_3)(\hat{i} \cdot \hat{k}) + (a_2 b_1)(\hat{j} \cdot \hat{i}) + (a_2 b_2)(\hat{j} \cdot \hat{j}) + (a_2 b_3)(\hat{j} \cdot \hat{k}) + (a_3 b_1)(\hat{k} \cdot \hat{i}) + (a_3 b_2)(\hat{k} \cdot \hat{j}) + (a_3 b_3)(\hat{k} \cdot \hat{k})$$

$$= (a_1 b_1 + a_2 b_2 + a_3 b_3)$$

$$[\because \hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1,\ \hat{i} \cdot \hat{j} = \hat{i} \cdot \hat{k} = \hat{j} \cdot \hat{i} = \ldots = 0].$$

Hence, $(a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}) \cdot (b_1 \hat{i} + b_2 \hat{j} + b_3 \hat{k}) = (a_1 b_1 + a_2 b_2 + a_3 b_3).$

CONDITION OF PERPENDICULARITY

Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$.

Then, $\vec{a} \perp \vec{b} \Leftrightarrow \vec{a} \cdot \vec{b} = 0 \Leftrightarrow a_1b_1 + a_2b_2 + a_3b_3 = 0.$

Thus, $\vec{a} \perp \vec{b} \Leftrightarrow a_1b_1 + a_2b_2 + a_3b_3 = 0.$

ANGLE BETWEEN TWO VECTORS

Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, and let θ be the angle between them. Then,

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta \Leftrightarrow a_1b_1 + a_2b_2 + a_3b_3 = |\vec{a}||\vec{b}|\cos\theta$$

$$\Leftrightarrow \cos\theta = \frac{a_1b_1 + a_2b_2 + a_3b_3}{|\vec{a}||\vec{b}|}$$

$$\Leftrightarrow \theta = \cos^{-1}\left\{\frac{a_1b_1 + a_2b_2 + a_3b_3}{\left(\sqrt{a_1^2 + a_2^2 + a_3^2}\right)\left(\sqrt{b_1^2 + b_2^2 + b_3^2}\right)}\right\}.$$

SOLVED EXAMPLES

EXAMPLE 1 *Find the projection of the vector* $(\hat{i} + 3\hat{j} + 7\hat{k})$ *on the vector* $(2\hat{i} - 3\hat{j} + 6\hat{k})$. **[CBSE 2014]**

SOLUTION Let $\vec{a} = (\hat{i} + 3\hat{j} + 7\hat{k})$ and $\vec{b} = (2\hat{i} - 3\hat{j} + 6\hat{k})$. Then,

$$\text{projection of } \vec{a} \text{ on } \vec{b} = \frac{(\vec{a} \cdot \vec{b})}{|\vec{b}|}$$

$$= \frac{(\hat{i} + 3\hat{j} + 7\hat{k}) \cdot (2\hat{i} - 3\hat{j} + 6\hat{k})}{\sqrt{4 + 9 + 36}}$$

$$= \frac{(2 - 9 + 42)}{\sqrt{49}} = \frac{35}{7} = 5.$$

EXAMPLE 2 *Write the projection of* $(\vec{b} + \vec{c})$ *on* $\vec{a}$, *where* $\vec{a} = (2\hat{i} - 2\hat{j} + \hat{k})$, $\vec{b} = (\hat{i} + 2\hat{j} - 2\hat{k})$ *and* $\vec{c} = (2\hat{i} - \hat{j} + 4\hat{k})$. **[CBSE 2013C]**

SOLUTION We have

$$(\vec{b} + \vec{c}) = (\hat{i} + 2\hat{j} - 2\hat{k}) + (2\hat{i} - \hat{j} + 4\hat{k})$$

$$= (1 + 2)\hat{i} + (2 - 1)\hat{j} + (-2 + 4)\hat{k} = (3\hat{i} + \hat{j} + 2\hat{k}).$$

$$\therefore \quad \text{projection of } (\vec{b}+\vec{c}) \text{ on } \vec{a} = \frac{(\vec{b}+\vec{c})\cdot\vec{a}}{|\vec{a}|}$$

$$= \frac{(3\hat{i}+\hat{j}+2\hat{k})\cdot(2\hat{i}-2\hat{j}+\hat{k})}{\sqrt{2^2+(-2)^2+1^2}}$$

$$= \frac{(6-2+2)}{\sqrt{9}} = \frac{6}{3} = 2.$$

EXAMPLE 3 *Find λ when the projection of $\vec{a} = \lambda\hat{i}+\hat{j}+4\hat{k}$ on $\vec{b} = (2\hat{i}+6\hat{j}+3\hat{k})$ is 4 units.* **[CBSE 2012]**

SOLUTION Projection of $\vec{a}$ on $\vec{b} = \frac{(\vec{a}\cdot\vec{b})}{|\vec{b}|}$

$$= \frac{(\lambda\hat{i}+\hat{j}+4\hat{k})\cdot(2\hat{i}+6\hat{j}+3\hat{k})}{\sqrt{2^2+6^2+3^2}}$$

$$= \frac{(2\lambda+6+12)}{\sqrt{4+36+9}} = \frac{(2\lambda+18)}{\sqrt{49}} = \frac{2(\lambda+9)}{7}.$$

$$\therefore \quad \frac{2(\lambda+9)}{7} = 4 \Rightarrow 2(\lambda+9) = 28 \Rightarrow \lambda+9 = 14 \Rightarrow \lambda = 5.$$

Hence, $\lambda = 5$.

EXAMPLE 4 *Write the value of λ so that the vectors $\vec{a} = 2\hat{i}+\lambda\hat{j}+\hat{k}$ and $\vec{b} = \hat{i}-2\hat{j}+3\hat{k}$ are perpendicular to each other.* **[CBSE 2013C]**

SOLUTION $\vec{a} \perp \vec{b} \Leftrightarrow \vec{a}\cdot\vec{b} = 0$

$$\Leftrightarrow (2\hat{i}+\lambda\hat{j}+\hat{k})\cdot(\hat{i}-2\hat{j}+3\hat{k}) = 0$$

$$\Leftrightarrow (2-2\lambda+3) = 0 \Leftrightarrow 2\lambda = 5 \Leftrightarrow \lambda = \frac{5}{2}.$$

Hence, $\lambda = \frac{5}{2}$.

EXAMPLE 5 *The scalar product of the vector $(\hat{i}+\hat{j}+\hat{k})$ with the unit vector along the sum of the vectors $(2\hat{i}+4\hat{j}-5\hat{k})$ and $(\lambda\hat{i}+2\hat{j}+3\hat{k})$ is equal to 1. Find the value of λ.* **[CBSE 2009, '14]**

SOLUTION Let $\vec{a} = (\hat{i}+\hat{j}+\hat{k})$, $\vec{b} = (2\hat{i}+4\hat{j}-5\hat{k})$ and $\vec{c} = (\lambda\hat{i}+2\hat{j}+3\hat{k})$.

Then,

$$(\vec{b}+\vec{c}) = (2\hat{i}+4\hat{j}-5\hat{k})+(\lambda\hat{i}+2\hat{j}+3\hat{k}) = (2+\lambda)\hat{i}+6\hat{j}-2\hat{k}.$$

$$\text{Unit vector along } (\vec{b}+\vec{c}) = \frac{(\vec{b}+\vec{c})}{|\vec{b}+\vec{c}|} = \frac{(2+\lambda)\hat{i}+6\hat{j}-2\hat{k}}{\sqrt{(2+\lambda)^2+6^2+(-2)^2}}$$

$$= \frac{(2+\lambda)\hat{i}+6\hat{j}-2\hat{k}}{\sqrt{\lambda^2+4\lambda+44}}.$$

But, $\frac{(\vec{b}+\vec{c})}{|\vec{b}+\vec{c}|}\cdot\vec{a} = 1$ (given).

$\therefore \quad \frac{(2+\lambda)\hat{i}+6\hat{j}-2\hat{k}}{\sqrt{\lambda^2+4\lambda+44}}\cdot(\hat{i}+\hat{j}+\hat{k}) = 1$

$\Rightarrow \quad (2+\lambda)\hat{i}+6\hat{j}-2\hat{k})\cdot(\hat{i}+\hat{j}+\hat{k}) = \sqrt{\lambda^2+4\lambda+44}$

$\Rightarrow \quad (2+\lambda)+6-2 = \sqrt{\lambda^2+4\lambda+44} \Rightarrow (6+\lambda) = \sqrt{\lambda^2+4\lambda+44}$

$\Rightarrow \quad \lambda^2+4\lambda+44 = (6+\lambda)^2 \Rightarrow \lambda^2+4\lambda+44 = 36+\lambda^2+12\lambda$

$\Rightarrow \quad 8\lambda = 8 \Rightarrow \lambda = 1.$

Hence, the required value of λ is 1.

EXAMPLE 6 *Dot products of a vector with the vectors* $(\hat{i}-\hat{j}+\hat{k}), (2\hat{i}+\hat{j}-3\hat{k})$ *and* $(\hat{i}+\hat{j}+\hat{k})$ *are respectively* 4, 0 *and* 2. *Find the vector.* **[CBSE 2013C]**

SOLUTION Let the required vector be $(x\hat{i}+y\hat{j}+z\hat{k})$. Then,

$$(x\hat{i}+y\hat{j}+z\hat{k})\cdot(\hat{i}-\hat{j}+\hat{k}) = 4 \Rightarrow x-y+z = 4 \quad \text{... (i)}$$

$$(x\hat{i}+y\hat{j}+z\hat{k})\cdot(2\hat{i}+\hat{j}-3\hat{k}) = 0 \Rightarrow 2x+y-3z = 0. \quad \text{... (ii)}$$

$$(x\hat{i}+y\hat{j}+z\hat{k})\cdot(\hat{i}+\hat{j}+\hat{k}) = 2 \Rightarrow x+y+z = 2. \quad \text{... (iii)}$$

On subtracting (i) from (iii) we get $2y = -2 \Rightarrow y = -1$.

On adding (i) and (ii), we get $3x-2z = 4$ ... (iv)

On adding (i) and (iii), we get $2x+2z = 6$ ... (v)

On solving (iv) and (v), we get $x = 2$ and $z = 1$.

$\therefore \quad x = 2, y = -1$ and $z = 1$.

Hence, the required vector is $(2\hat{i}-\hat{j}+\hat{k})$.

EXAMPLE 7 *Let* $\vec{a} = \hat{i}+4\hat{j}+2\hat{k}$, $\vec{b} = 3\hat{i}-2\hat{j}+7\hat{k}$ *and* $\vec{c} = 2\hat{i}-\hat{j}+4\hat{k}$. *Find a vector* $\vec{p}$ *which is perpendicualr to both* $\vec{a}$ *and* $\vec{b}$ *and* $\vec{p}\cdot\vec{c} = 18$.

[CBSE 2012]

SOLUTION Let $\vec{p} = (x\hat{i}+y\hat{j}+z\hat{k})$. Then,

$\vec{p} \perp \vec{a}$, $\vec{p} \perp \vec{b}$ and $\vec{p}\cdot\vec{c} = 18$

$\Rightarrow \quad \vec{p}\cdot\vec{a} = 0, \vec{p}\cdot\vec{b} = 0$ and $\vec{p}\cdot\vec{c} = 18$

$$\Rightarrow \begin{cases} (x\hat{i}+y\hat{j}+z\hat{k})\cdot(\hat{i}+4\hat{j}+2\hat{k})=0 \Rightarrow x+4y+2z=0 & \text{... (i)} \\ (x\hat{i}+y\hat{j}+z\hat{k})\cdot(3\hat{i}-2\hat{j}+7\hat{k})=0 \Rightarrow 3x-2y+7z=0 & \text{... (ii)} \\ (x\hat{i}+y\hat{j}+z\hat{k})\cdot(2\hat{i}-\hat{j}+4\hat{k})=18 \Rightarrow 2x-y+4z=18 & \text{... (iii)} \end{cases}$$

On solving (i) and (ii) by cross multiplication, we get

$$\frac{x}{(28+4)}=\frac{y}{(6-7)}=\frac{z}{(-2-12)}=k \text{ (say)}$$

$\Rightarrow \quad x=32k,\ y=-k$ and $z=-14k.$

Substituting these values in (iii), we get:

$64k+k-56k=18 \Rightarrow 9k=18 \Rightarrow k=2.$

$\therefore \quad x=(32\times 2)=64,\ y=-2$ and $z=(-14)\times 2=-28.$

Hence, the required vector is $(64\hat{i}-2\hat{j}-28\hat{k}).$

EXAMPLE 8 *Find a vector whose magnitude is 3 units and which is perpendicular to each of the vectors* $\vec{a}=3\hat{i}+\hat{j}-4\hat{k}$ *and* $\vec{b}=6\hat{i}+5\hat{j}-2\hat{k}.$ **[CBSE 2000C]**

SOLUTION Let the required vector be $\vec{c}=c_1\hat{i}+c_2\hat{j}+c_3\hat{k}.$

Then, $|\vec{c}|=3 \Leftrightarrow \sqrt{c_1^2+c_2^2+c_3^2}=3 \Leftrightarrow c_1^2+c_2^2+c_3^2=9.$ … (i)

Also, $\vec{c}\perp\vec{a} \Rightarrow \vec{c}\cdot\vec{a}=0$

$$\Rightarrow (c_1\hat{i}+c_2\hat{j}+c_3\hat{k})\cdot(3\hat{i}+\hat{j}-4\hat{k})=0$$

$$\Rightarrow 3c_1+c_2-4c_3=0. \qquad \text{… (ii)}$$

And, $\vec{c}\perp\vec{b} \Rightarrow \vec{c}\cdot\vec{b}=0$

$$\Rightarrow (c_1\hat{i}+c_2\hat{j}+c_3\hat{k})\cdot(6\hat{i}+5\hat{j}-2\hat{k})=0$$

$$\Rightarrow 6c_1+5c_2-2c_3=0. \qquad \text{… (iii)}$$

From (ii) and (iii), by cross multiplication, we get

$$\frac{c_1}{(-2+20)}=\frac{c_2}{(-24+6)}=\frac{c_3}{(15-6)}=k \text{ (say)}$$

$$\Rightarrow \frac{c_1}{18}=\frac{c_2}{-18}=\frac{c_3}{9}=k \Rightarrow \frac{c_1}{2}=\frac{c_2}{-2}=\frac{c_3}{1}=k$$

$\Rightarrow \quad c_1=2k,\ c_2=-2k$ and $c_3=k.$

Substituting these values in (i), we get

$4k^2+4k^2+k^2=9 \Rightarrow k^2=1 \Rightarrow k=1.$

$\therefore \quad c_1=2,\ c_2=-2$ and $c_3=1.$

Hence, $\vec{c}=(2\hat{i}-2\hat{j}+\hat{k})$ is the required vector.

EXAMPLE 9 *If* $\vec{a}$ *and* $\vec{b}$ *are vectors such that* $|\vec{a}| = 2, |\vec{b}| = 3$ *and* $\vec{a} \cdot \vec{b} = 4$, *find* $|\vec{a} - \vec{b}|$.

SOLUTION We have

$$|\vec{a} - \vec{b}|^2 = (\vec{a} - \vec{b}) \cdot (\vec{a} - \vec{b})$$
$$= \vec{a} \cdot \vec{a} - \vec{a} \cdot \vec{b} - \vec{b} \cdot \vec{a} + \vec{b} \cdot \vec{b} \quad \text{(by distributive law)}$$
$$= |\vec{a}|^2 - 2\,\vec{a} \cdot \vec{b} + |\vec{b}|^2 \qquad [\because\ \vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}\ ]$$
$$= (2^2 - 2 \times 4 + 3^2) = 5.$$

Hence, $|\vec{a} - \vec{b}| = \sqrt{5}$.

EXAMPLE 10 *If* $\vec{a}$ *makes equal angles with the coordinate axes and has magnitude* 3, *find the angle between* $\vec{a}$ *and each of the three coordinate axes.*

SOLUTION Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and let α be the angle between $\vec{a}$ and each of the coordinate axes.

Then, α is the angle between $\vec{a}$ and each one of $\hat{i}$, $\hat{j}$ and $\hat{k}$.

$$\therefore \quad \cos\alpha = \frac{\vec{a} \cdot \hat{i}}{|\vec{a}|\,|\hat{i}|} = \frac{a_1}{3} \Rightarrow a_1 = 3\cos\alpha \ [\because\ \vec{a} \cdot \hat{i} = a_1, |\vec{a}| = 3, |\hat{i}| = 1].$$

Similarly, $a_2 = 3\cos\alpha$ and $a_3 = 3\cos\alpha$.

Now, $|\vec{a}| = 3 \Rightarrow |\vec{a}|^2 = 9$

$$\Rightarrow a_1^2 + a_2^2 + a_3^2 = 9$$
$$\Rightarrow 9\cos^2\alpha + 9\cos^2\alpha + 9\cos^2\alpha = 9$$
$$\Rightarrow 27\cos^2\alpha = 9 \Rightarrow \cos^2\alpha = \frac{1}{3}$$
$$\Rightarrow \cos\alpha = \frac{1}{\sqrt{3}} \Rightarrow \alpha = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

Hence, the required angle is $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$.

EXAMPLE 11 *If a unit vector* $\vec{a}$ *makes angles* $\pi/4$ *with* $\hat{i}$, $\pi/3$ *with* $\hat{j}$ *and an acute angle* θ *with* $\hat{k}$ *then find the value of* θ. *Also, find the scalar and vector components of* $\vec{a}$ *along the axes.* **[CBSE 2013]**

SOLUTION Let $\vec{a} = (a_1\hat{i} + a_2\hat{j} + a_3\hat{k})$. Then, $\vec{a}$ being a unit vector, we have $(a_1^2 + a_2^2 + a_3^2) = 1$.

Now, $\vec{a}\cdot\hat{i}=a_1 \Rightarrow |\vec{a}||\hat{i}|\cos\frac{\pi}{4}=a_1 \Rightarrow a_1=\frac{1}{\sqrt{2}}$ $[\because |\vec{a}|=1, |\hat{i}|=1]$

$\vec{a}\cdot\hat{j}=a_2 \Rightarrow |\vec{a}||\hat{j}|\cos\frac{\pi}{3}=a_2 \Rightarrow a_2=\frac{1}{2}$ $[\because |\vec{a}|=1, |\hat{j}|=1]$

$\vec{a}\cdot\hat{k}=a_3 \Rightarrow |\vec{a}||\hat{k}|\cos\theta=a_3 \Rightarrow a_3=\cos\theta [\because |\vec{a}|=1, |\hat{k}|=1]$

Now, $|\vec{a}|=1 \Rightarrow |\vec{a}|^2=1$

$\Rightarrow a_1{}^2+a_2{}^2+a_3{}^2=1$

$\Rightarrow \frac{1}{2}+\frac{1}{4}+\cos^2\theta=1$ $[\because a_1=\frac{1}{\sqrt{2}}, a_2=\frac{1}{2}, a_3=\cos\theta]$

$\Rightarrow \cos^2\theta=\frac{1}{4} \Rightarrow \cos\theta=\frac{1}{2} \Rightarrow \theta=\frac{\pi}{3}.$

Hence, the scalar componets of $\vec{a}$ are $\frac{1}{\sqrt{2}}, \frac{1}{2}$ and $\frac{1}{2}$.

And, the vector components of $\vec{a}$ are $\frac{1}{\sqrt{2}}\hat{i}, \frac{1}{2}\hat{j}$ and $\frac{1}{2}\hat{k}$.

EXAMPLE 12 *If $\vec{a}$ and $\vec{b}$ are two unit vectors such that $\vec{a}+\vec{b}$ is also a unit vector, then find the angle between $\vec{a}$ and $\vec{b}$.* **[CBSE 2014]**

SOLUTION Let 0 be the angle between the unit vectors $\vec{a}$ and $\vec{b}$.

Since $\vec{a}$ and $\vec{b}$ are unit vectors, we have $|\vec{a}|=1$ and $|\vec{b}|=1$.

Again, since $(\vec{a}+\vec{b})$ is a unit vector, we have

$|\vec{a}+\vec{b}|^2=1$

$\Rightarrow (\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b})=1 \Rightarrow \vec{a}\cdot\vec{a}+\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{a}+\vec{b}\cdot\vec{b}=1$

$\Rightarrow |\vec{a}|^2+2(\vec{a}\cdot\vec{b})+|\vec{b}|^2=1$ $[\because \vec{b}\cdot\vec{a}=\vec{a}\cdot\vec{b}]$

$\Rightarrow 2(\vec{a}\cdot\vec{b})=-1 \Rightarrow \vec{a}\cdot\vec{b}=\frac{-1}{2}$ $[\because |\vec{a}|^2=1, |\vec{b}|^2=1]$

$\Rightarrow |\vec{a}||\vec{b}|\cos\theta=\frac{-1}{2} \Rightarrow \cos\theta=\frac{-1}{2} \Rightarrow \theta=\frac{2\pi}{3}$ $[\because |\vec{a}|=1, |\vec{b}|=1].$

Hence, the angle between $\vec{a}$ and $\vec{b}$ is $\frac{2\pi}{3}$.

EXAMPLE 13 *If the sum of two unit vectors $\hat{a}$ and $\hat{b}$ is a unit vector, show that the magnitude of their difference is $\sqrt{3}$.* **[CBSE 2012C]**

SOLUTION Let $\hat{a}+\hat{b}=\hat{c}$, where $\hat{c}$ is a unit vector. Then,

$\hat{a}+\hat{b}=\hat{c} \Rightarrow (\hat{a}+\hat{b})\cdot(\hat{a}+\hat{b})=\hat{c}\cdot\hat{c}$

$\Rightarrow \hat{a}\cdot\hat{a}+\hat{a}\cdot\hat{b}+\hat{b}\cdot\hat{a}+\hat{b}\cdot\hat{b}=\hat{c}\cdot\hat{c}$

$\Rightarrow |\hat{a}|^2+2(\hat{a}\cdot\hat{b})+|\hat{b}|^2=|\hat{c}|^2$ $[\because \hat{b}\cdot\hat{a}=\hat{a}\cdot\hat{b}]$

$\Rightarrow 1+2(\hat{a}\cdot\hat{b})+1=1$ $[\because |\hat{a}|=|\hat{b}|=|\hat{c}|=1]$

$\Rightarrow 2(\hat{a}\cdot\hat{b})=-1$ ... (i)

Now, $|\hat{a}-\hat{b}|^2=(\hat{a}-\hat{b})\cdot(\hat{a}-\hat{b})$

$=\hat{a}\cdot\hat{a}-\hat{a}\cdot\hat{b}-\hat{b}\cdot\hat{a}+\hat{b}\cdot\hat{b}$

$=|\hat{a}|^2-2(\hat{a}\cdot\hat{b})+|\hat{b}|^2$ $[\because \hat{b}\cdot\hat{a}=\hat{a}\cdot\hat{b}]$

$=1-2(\hat{a}\cdot\hat{b})+1=1+1+1=3$ [using (i)].

Hence, $|\hat{a}-\hat{b}|=\sqrt{3}$.

EXAMPLE 14 *If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are three vectors such that $|\vec{a}|=5$, $|\vec{b}|=12$, $|\vec{c}|=13$ and $\vec{a}+\vec{b}+\vec{c}=0$, find the value of $(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a})$.* **[CBSE 2012]**

SOLUTION $\vec{a}+\vec{b}+\vec{c}=0 \Rightarrow \vec{a}+\vec{b}=-\vec{c}$

$\Rightarrow (\vec{a}+\vec{b})\cdot\vec{c}=(-\vec{c})\cdot\vec{c} \Rightarrow \vec{a}\cdot\vec{c}+\vec{b}\cdot\vec{c}=-|\vec{c}|^2$

$\Rightarrow \vec{c}\cdot\vec{a}+\vec{b}\cdot\vec{c}=-169$ ... (i)

$[\because \vec{a}\cdot\vec{c}=\vec{c}\cdot\vec{a}$ and $|\vec{c}|=13]$

Again, $\vec{a}+\vec{b}+\vec{c}=0 \Rightarrow \vec{b}+\vec{c}=-\vec{a}$

$\Rightarrow (\vec{b}+\vec{c})\cdot\vec{a}=(-\vec{a})\cdot\vec{a}$

$\Rightarrow \vec{b}\cdot\vec{a}+\vec{c}\cdot\vec{a}=-|\vec{a}|^2$

$\Rightarrow \vec{a}\cdot\vec{b}+\vec{c}\cdot\vec{a}=-25$ ... (ii)

$[\because \vec{b}\cdot\vec{a}=\vec{a}\cdot\vec{b}$ and $|\vec{a}|=5]$

Also, $\vec{a}+\vec{b}+\vec{c}=0 \Rightarrow \vec{a}+\vec{c}=-\vec{b}$

$\Rightarrow (\vec{a}+\vec{c})\cdot\vec{b}=(-\vec{b})\cdot\vec{b}$

$\Rightarrow \vec{a}\cdot\vec{b}+\vec{c}\cdot\vec{b}=-|\vec{b}|^2$

$\Rightarrow \vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}=-144$ ... (iii)

$[\because \vec{c}\cdot\vec{b}=\vec{b}\cdot\vec{c}$ and $|\vec{b}|=12]$

Adding the corresponding sides of (i), (ii) and (iii), we get

$$2(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}) = -(169+25+144)$$

$$\Rightarrow \quad (\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}) = \frac{-338}{2} = -169.$$

Hence, $(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}) = -169.$

EXAMPLE 15 *If $\vec{a}, \vec{b}, \vec{c}$ are three vectors such that $|\vec{a}| = 3, |\vec{b}| = 4$ and $|\vec{c}| = 5$ and each one of them is perpendicular to the sum of the two then find $|\vec{a}+\vec{b}+\vec{c}|$.* **[CBSE 2011C]**

SOLUTION Let $\vec{a}, \vec{b}, \vec{c}$ be the given vectors such that

$$\{|\vec{a}| = 3, |\vec{b}| = 4, |\vec{c}| = 5\}, \qquad \text{... (i)}$$

$$\left.\begin{aligned}\vec{a}\cdot(\vec{b}+\vec{c}) &= 0\\ \vec{b}\cdot(\vec{c}+\vec{a}) &= 0\\ \vec{c}\cdot(\vec{a}+\vec{b}) &= 0\end{aligned}\right\}. \qquad \text{... (ii)}$$

$$\therefore \quad |\vec{a}+\vec{b}+\vec{c}|^2 = (\vec{a}+\vec{b}+\vec{c})\cdot(\vec{a}+\vec{b}+\vec{c})$$

$$= \vec{a}\cdot\vec{a}+\vec{a}\cdot(\vec{b}+\vec{c})+\vec{b}\cdot(\vec{c}+\vec{a})+\vec{b}\cdot\vec{b}$$

$$+\vec{c}\cdot(\vec{a}+\vec{b})+\vec{c}\cdot\vec{c}$$

$$= |\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2 \quad \text{[using (ii)]}$$

$$= (3^2+4^2+5^2) = (9+16+25) = 50$$

Hence, $|\vec{a}+\vec{b}+\vec{c}| = \sqrt{50} = 5\sqrt{2}.$

EXAMPLE 16 *If $\vec{a}, \vec{b}$ and $\vec{c}$ are three mutually perpendicular vectors of the same magnitude, prove that $(\vec{a}+\vec{b}+\vec{c})$ is equally inclined to the vectors $\vec{a}, \vec{b}$ and $\vec{c}$. Also find this angle.* **[CBSE 2006C, '11C, '13C]**

SOLUTION It is given that

$$|\vec{a}| = |\vec{b}| = |\vec{c}| = a \text{ (say)} \qquad \text{... (i)}$$

Since $\vec{a}, \vec{b}$ and $\vec{c}$ are mutually perpendicular vectors, we have

$$\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{c} = \vec{c}\cdot\vec{a} = 0. \qquad \text{... (ii)}$$

Now, $|\vec{a}+\vec{b}+\vec{c}|^2 = (\vec{a}+\vec{b}+\vec{c})\cdot(\vec{a}+\vec{b}+\vec{c})$

$$= \vec{a}\cdot\vec{a} + \vec{b}\cdot\vec{b} + \vec{c}\cdot\vec{c} + 2(\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a})$$

$$= |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 \text{ [using (ii)]}$$

$$= 3a^2 \text{ [using (i)].}$$

$$\therefore \quad |\vec{a} + \vec{b} + \vec{c}| = \sqrt{3}a. \qquad \text{... (iii)}$$

Let $(\vec{a} + \vec{b} + \vec{c})$ make angles α, β and γ with $\vec{a}$, $\vec{b}$ and $\vec{c}$ respectively.

Then $(\vec{a} + \vec{b} + \vec{c})\cdot\vec{a} = |\vec{a} + \vec{b} + \vec{c}|\,|\vec{a}|\cos\alpha$

$$= (\sqrt{3}a \times a \times \cos\alpha) = \sqrt{3}a^2\cos\alpha$$

$$\Rightarrow \quad (\vec{a}\cdot\vec{a} + \vec{b}\cdot\vec{a} + \vec{c}\cdot\vec{a}) = \sqrt{3}a^2\cos\alpha$$

$$\Rightarrow \quad |\vec{a}|^2 = \sqrt{3}a^2\cos\alpha \;\Rightarrow\; a^2 = \sqrt{3}a^2\cos\alpha$$

$$\Rightarrow \quad \cos\alpha = \frac{1}{\sqrt{3}} \;\Rightarrow\; \alpha = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right).$$

Similarly, $\beta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$ and $\gamma = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$.

$$\therefore \quad \alpha = \beta = \gamma = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right).$$

Hence, $(\vec{a} + \vec{b} + \vec{c})$ is equally inclined to $\vec{a}$, $\vec{b}$ and $\vec{c}$ and the required angle is $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$.

EXAMPLE 17 *If $\vec{a}, \vec{b}, \vec{c}$ are unit vectors such that $\vec{a} + \vec{b} + \vec{c} = \vec{0}$ then find the value of $(\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a})$.*

SOLUTION Since $\vec{a}, \vec{b}, \vec{c}$ are unit vectors, we have

$$|\vec{a}| = 1, |\vec{b}| = 1 \text{ and } |\vec{c}| = 1.$$

Now, $\vec{a} + \vec{b} + \vec{c} = \vec{0}$

$$\Rightarrow \quad (\vec{a} + \vec{b} + \vec{c})\cdot(\vec{a} + \vec{b} + \vec{c}) = 0 \qquad [\because\ \vec{0}\cdot\vec{0} = 0]$$

$$\Rightarrow \quad \vec{a}\cdot\vec{a} + \vec{b}\cdot\vec{b} + \vec{c}\cdot\vec{c} + 2(\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a}) = 0$$

$$\Rightarrow \quad |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 2(\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a}) = 0$$

$$\Rightarrow \quad (\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a}) = -\frac{3}{2} \quad [\because\ |\vec{a}|^2 = 1, |\vec{b}|^2 = 1, |\vec{c}|^2 = 1].$$

Hence, $(\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a}) = -\frac{3}{2}$.

EXAMPLE 18 *If* $\vec{a}\cdot\vec{b}=\vec{a}\cdot\vec{c}$, *show that* $\vec{a}=\vec{0}$ *or* $\vec{b}=\vec{c}$ *or* $\vec{a}\perp(\vec{b}-\vec{c})$.

SOLUTION $\vec{a}\cdot\vec{b}=\vec{a}\cdot\vec{c} \Rightarrow \vec{a}\cdot\vec{b}-\vec{a}\cdot\vec{c}=0 \Rightarrow \vec{a}\cdot(\vec{b}-\vec{c})=0$

$\Rightarrow \vec{a}=\vec{0}$ or $(\vec{b}-\vec{c})=\vec{0}$ or $\vec{a}\perp(\vec{b}-\vec{c})$

$\Rightarrow \vec{a}=\vec{0}$ or $\vec{b}=\vec{c}$ or $\vec{a}\perp(\vec{b}-\vec{c})$.

Hence, $\vec{a}\cdot\vec{b}=\vec{a}\cdot\vec{c} \Rightarrow \vec{a}=\vec{0}$ or $\vec{b}=\vec{c}$ or $\vec{a}\perp(\vec{b}-\vec{c})$.

EXAMPLE 19 *Let* $\vec{a}$ *and* $\vec{b}$ *be two nonzero vectors. Prove that*

$$\vec{a}\perp\vec{b} \Leftrightarrow |\vec{a}+\vec{b}|=|\vec{a}-\vec{b}|.$$

SOLUTION Let $\vec{a}\perp\vec{b}$. Then, $(\vec{a}\cdot\vec{b})=0$. … (i)

Now, $|\vec{a}+\vec{b}|^2=(\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b})$

$=\vec{a}\cdot\vec{a}+\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{a}+\vec{b}\cdot\vec{b}$

$=|\vec{a}|^2+|\vec{b}|^2$ $\{\because\ \vec{a}\cdot\vec{b}=0$ and $\vec{b}\cdot\vec{a}=\vec{a}\cdot\vec{b}=0\}$.

Also, $|\vec{a}-\vec{b}|^2=(\vec{a}-\vec{b})\cdot(\vec{a}-\vec{b})$

$=\vec{a}\cdot\vec{a}-\vec{a}\cdot\vec{b}-\vec{b}\cdot\vec{a}+\vec{b}\cdot\vec{b}$

$=|\vec{a}|^2+|\vec{b}|^2$ $\{\because\ \vec{a}\cdot\vec{b}=0$ and $\vec{b}\cdot\vec{a}=\vec{a}\cdot\vec{b}=0\}$.

Thus, $|\vec{a}+\vec{b}|^2=|\vec{a}-\vec{b}|^2$, and therefore, $|\vec{a}+\vec{b}|=|\vec{a}-\vec{b}|$.

$\therefore\ \vec{a}\perp\vec{b} \Rightarrow |\vec{a}+\vec{b}|=|\vec{a}-\vec{b}|$.

Conversely, suppose that $|\vec{a}+\vec{b}|=|\vec{a}-\vec{b}|$. Then,

$|\vec{a}+\vec{b}|=|\vec{a}-\vec{b}| \Rightarrow |\vec{a}+\vec{b}|^2=|\vec{a}-\vec{b}|^2$

$\Rightarrow (\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b})=(\vec{a}-\vec{b})\cdot(\vec{a}-\vec{b})$

$\Rightarrow \vec{a}\cdot\vec{a}+\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{a}+\vec{b}\cdot\vec{b}$

$=\vec{a}\cdot\vec{a}-\vec{a}\cdot\vec{b}-\vec{b}\cdot\vec{a}+\vec{b}\cdot\vec{b}$

$\Rightarrow 2(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{a})=0$

$\Rightarrow 4(\vec{a}\cdot\vec{b})=0$ $[\because\ \vec{b}\cdot\vec{a}=\vec{a}\cdot\vec{b}]$

$\Rightarrow \vec{a}\cdot\vec{b}=0 \Rightarrow \vec{a}\perp\vec{b}$.

Thus, $|\vec{a}+\vec{b}|=|\vec{a}-\vec{b}| \Rightarrow \vec{a}\perp\vec{b}$.

Hence, $|\vec{a}+\vec{b}|=|\vec{a}-\vec{b}| \Leftrightarrow \vec{a}\perp\vec{b}$.

EXAMPLE 20 *Express the vector* $\vec{a} = (5\hat{i} - 2\hat{j} + 5\hat{k})$ *as sum of two vectors such that one is parallel to the vector* $\vec{b} = (3\hat{i} + \hat{k})$ *and the other is perpendicular to* $\vec{b}$. **[CBSE 2005]**

SOLUTION Any vector parallel to $\vec{b}$ is of the form $\lambda\,\vec{b}$ for some scalar λ.

Let $\vec{a} = \lambda\,\vec{b} + \vec{c}$, where $\vec{c} \perp \vec{b}$. Then, $\vec{c} = (\vec{a} - \lambda\,\vec{b}) \perp \vec{b}$

$\Leftrightarrow\ (\vec{a} - \lambda\,\vec{b}) \cdot \vec{b} = 0 \Leftrightarrow\ (\vec{a} \cdot \vec{b}) - \lambda(\vec{b} \cdot \vec{b}) = 0$

$\Leftrightarrow\ (5\hat{i} - 2\hat{j} + 5\hat{k}) \cdot (3\hat{i} + \hat{k}) - \lambda(3\hat{i} + \hat{k}) \cdot (3\hat{i} + \hat{k}) = 0$

$\Leftrightarrow\ (15 - 0 + 5) - \lambda(9 + 1) = 0 \Leftrightarrow\ 10\lambda = 20 \Leftrightarrow\ \lambda = 2.$

$\therefore\quad \lambda\,\vec{b} = 2\,\vec{b} = (6\hat{i} + 2\hat{k}).$

And, $\vec{c} = (\vec{a} - 2\vec{b}) = (5\hat{i} - 2\hat{j} + 5\hat{k}) - 2(3\hat{i} + \hat{k}) = (-\hat{i} - 2\hat{j} + 3\hat{k}).$

Hence, the required vectors are $(6\hat{i} + 2\hat{k})$ and $(-\hat{i} - 2\hat{j} + 3\hat{k})$.

EXAMPLE 21 *Find the values of* λ *for which the angle between the vectors* $\vec{a} = 2\lambda^2\hat{i} + 4\lambda\hat{j} + \hat{k}$ *and* $\vec{b} = 7\hat{i} - 2\hat{j} + \lambda\hat{k}$ *is obtuse.* **[CBSE 2013C]**

SOLUTION Let θ be the angle between $\vec{a}$ and $\vec{b}$. Then,

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|\,|\vec{b}|}. \qquad \ldots (i)$$

Clearly, θ is obtuse $\Leftrightarrow\ \cos\theta < 0 \Leftrightarrow\ \vec{a} \cdot \vec{b} < 0$ [from (i)]

$\Leftrightarrow\ 14\lambda^2 - 8\lambda + \lambda < 0 \Leftrightarrow\ 14\lambda^2 - 7\lambda < 0$

$\Leftrightarrow\ 2\lambda^2 - \lambda < 0 \Leftrightarrow\ \lambda(2\lambda - 1) < 0.$

Now, $\lambda(2\lambda - 1) < 0 \Rightarrow$ either $\{\lambda < 0$ and $(2\lambda - 1) > 0\}$

or $\{\lambda > 0$ and $(2\lambda - 1) < 0\}$

$\Rightarrow \left\{\lambda < 0 \text{ and } \lambda > \frac{1}{2}\right\}$ or $\left\{\lambda > 0 \text{ and } \lambda < \frac{1}{2}\right\}$

$\Rightarrow \left\{\frac{1}{2} < \lambda < 0\right\}$ or $\left\{0 < \lambda < \frac{1}{2}\right\}$

$\Rightarrow\ 0 < \lambda < \frac{1}{2}\ \left[\because\ \frac{1}{2} < \lambda < 0 \text{ is not possible}\right]$

$\Rightarrow\ \lambda \in \left]0, \frac{1}{2}\right[.$

$\therefore$ the required values of λ are all real values in $\left]0, \frac{1}{2}\right[$.

EXAMPLE 22 *Let $A(0, 1, 1)$, $B(3, 1, 5)$ and $C(0, 3, 3)$ be the vertices of a $\triangle ABC$. Using vectors, show that $\triangle ABC$ is right angled at C.*

SOLUTION In order to show that $\overrightarrow{CA} \perp \overrightarrow{CB}$, we must show that $\overrightarrow{CA} \cdot \overrightarrow{CB} = 0$.

Let O be the origin. Then,

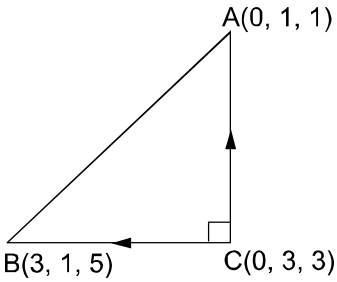
$$\overrightarrow{OA} = 0\hat{i} + \hat{j} + \hat{k},\ \overrightarrow{OB} = 3\hat{i} + \hat{j} + 5\hat{k} \text{ and } \overrightarrow{OC} = 0\hat{i} + 3\hat{j} + 3\hat{k}.$$

$$\therefore \quad \overrightarrow{CA} = (\overrightarrow{OA} - \overrightarrow{OC})$$

$$= (0\hat{i} + \hat{j} + \hat{k}) - (0\hat{i} + 3\hat{j} + 3\hat{k})$$

$$= (0 - 0)\hat{i} + (1 - 3)\hat{j} + (1 - 3)\hat{k}$$

$$= (0\hat{i} - 2\hat{j} - 2\hat{k}).$$



$$\therefore \quad \overrightarrow{CB} = (\overrightarrow{OB} - \overrightarrow{OC})$$

$$= (3\hat{i} + \hat{j} + 5\hat{k}) - (0\hat{i} + 3\hat{j} + 3\hat{k})$$

$$= (3\hat{i} - 2\hat{j} + 2\hat{k}).$$

$$\therefore \quad \overrightarrow{CA} \cdot \overrightarrow{CB} = (0\hat{i} - 2\hat{j} - 2\hat{k}) \cdot (3\hat{i} - 2\hat{j} + 2\hat{k})$$

$$= [(0 \times 3) + (-2) \times (-2) + (-2) \times 2] = 0$$

$$\Rightarrow \quad \overrightarrow{CA} \perp \overrightarrow{CB}.$$

Hence, $\triangle ABC$ is right angled at C.

EXAMPLE 23 *Show that the points A, B and C having position vectors $(2\hat{i} - \hat{j} + \hat{k})$, $(\hat{i} - 3\hat{j} - 5\hat{k})$ and $(3\hat{i} - 4\hat{j} - 4\hat{k})$ respectively are the vertices of a right-angled triangle. Also, find the remaining angles of the triangle.*

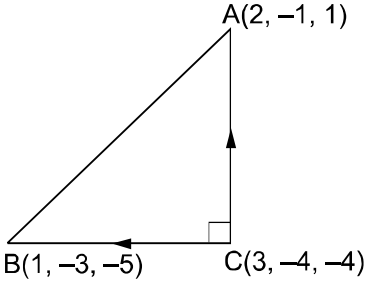
SOLUTION We have

$$\overrightarrow{AB} = (\text{p.v. of } B) - (\text{p.v. of } A)$$

$$= (-\hat{i} - 2\hat{j} - 6\hat{k}),$$

$$\overrightarrow{BC} = (\text{p.v. of } C) - (\text{p.v. of } B)$$

$$= (2\hat{i} - \hat{j} + \hat{k}) \text{ and}$$



$$\overrightarrow{CA} = (\text{p.v. of } A) - (\text{p.v. of } C) = (-\hat{i} + 3\hat{j} + 5\hat{k}).$$

Clearly, we have $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \vec{0}$.

$\therefore$ A, B, and C are the vertices of a triangle.

Now, $\overrightarrow{BC} \cdot \overrightarrow{CA} = (2\hat{i} - \hat{j} + \hat{k}) \cdot (-\hat{i} + 3\hat{j} + 5\hat{k}) = (-2 - 3 + 5) = 0.$

$\therefore$ $\overrightarrow{BC} \perp \overrightarrow{CA}$ and therefore, $\angle C = 90°$.

Now, $\angle A$ is the angle between $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

$$\therefore \quad \overrightarrow{AB}\cdot\overrightarrow{AC}=|\overrightarrow{AB}||\overrightarrow{AC}|\cos A \Rightarrow \cos A=\frac{(\overrightarrow{AB}\cdot\overrightarrow{AC})}{|\overrightarrow{AB}||\overrightarrow{AC}|}$$

$$=\frac{(-\hat{i}-2\hat{j}-6\hat{k})\cdot(\hat{i}-3\hat{j}-5\hat{k})}{\{\sqrt{(-1)^2+(-2)^2+(-6)^2}\}\cdot\{\sqrt{1^2+(-3)^2+(-5)^2}\}}$$

$$[\because \quad \overrightarrow{AC}=-\overrightarrow{CA}]$$

$$=\frac{(-1+6+30)}{\sqrt{41}\times\sqrt{35}}=\frac{35}{\sqrt{41}\times\sqrt{35}}=\frac{\sqrt{35}}{\sqrt{41}}=\sqrt{\frac{35}{41}}$$

$$\Rightarrow \quad A=\cos^{-1}\sqrt{\frac{35}{41}}.$$

Further, $\angle B$ is the angle between $\overrightarrow{BC}$ and $\overrightarrow{BA}$.

$$\therefore \quad \overrightarrow{BC}\cdot\overrightarrow{BA}=|\overrightarrow{BC}||\overrightarrow{BA}|\cos B$$

$$\Rightarrow \quad \cos B=\frac{(\overrightarrow{BC}\cdot\overrightarrow{BA})}{|\overrightarrow{BC}||\overrightarrow{BA}|}$$

$$=\frac{(2\hat{i}-\hat{j}+\hat{k})\cdot(\hat{i}+2\hat{j}+6\hat{k})}{\{\sqrt{2^2+(-1)^2+1^2}\}\cdot\{\sqrt{1^2+2^2+6^2}\}}$$

$$=\frac{(2-2+6)}{\sqrt{6}\times\sqrt{41}}=\frac{6}{\sqrt{6}\times\sqrt{41}}=\frac{\sqrt{6}}{\sqrt{41}}=\sqrt{\frac{6}{41}}.$$

$$\Rightarrow \quad B=\cos^{-1}\sqrt{\frac{6}{41}}.$$

$$\therefore \quad A=\cos^{-1}\sqrt{\frac{35}{41}},\ B=\cos^{-1}\sqrt{\frac{6}{41}},\ \text{and}\ C=90^\circ.$$

EXAMPLE 24 *Let* $(\hat{i}+\hat{j}+\hat{k})$, $(2\hat{i}+5\hat{j})$, $(3\hat{i}+2\hat{j}-3\hat{k})$ *and* $(\hat{i}-6\hat{j}-\hat{k})$ *be the position vectors of points A, B, C, D respectively. Find the angle between AB and CD. Hence, show that* $AB \,||\, CD$. **[CBSE 2008]**

SOLUTION Let the angle between $\overrightarrow{AB}$ and $\overrightarrow{CD}$ be θ.

Now, $\overrightarrow{AB}=(\text{p.v. of } B)-(\text{p.v. of } A)$

$$=(2\hat{i}+5\hat{j})-(\hat{i}+\hat{j}+\hat{k})=(\hat{i}+4\hat{j}-\hat{k})$$

and, $\overrightarrow{CD}=(\text{p.v. of } D)-(\text{p.v. of } C)$

$$=(\hat{i}-6\hat{j}-\hat{k})-(3\hat{i}+2\hat{j}-3\hat{k})=(-2\hat{i}-8\hat{j}+2\hat{k}).$$

$$\therefore \quad |\overrightarrow{AB}| = \sqrt{1^2 + 4^2 + (-1)^2} = \sqrt{18} = 3\sqrt{2}$$

and, $|\overrightarrow{CD}| = \sqrt{(-2)^2 + (-8)^2 + 2^2} = \sqrt{72} = 6\sqrt{2}.$

Now, $\overrightarrow{AB} \cdot \overrightarrow{CD} = (\hat{i} + 4\hat{j} - \hat{k}) \cdot (-2\hat{i} - 8\hat{j} + 2\hat{k})$

$= (-2 - 32 - 2) = -36.$

$$\therefore \quad \cos\theta = \frac{\overrightarrow{AB} \cdot \overrightarrow{CD}}{|\overrightarrow{AB}||\overrightarrow{CD}|} = \frac{-36}{(3\sqrt{2} \times 6\sqrt{2})} = \frac{-36}{36} = -1 \Rightarrow \theta = \pi.$$

Hence, $AB \,||\, CD$.

EXERCISE 23

1. Find $\vec{a} \cdot \vec{b}$ when

   (i) $\vec{a} = \hat{i} - 2\hat{j} + \hat{k}$ and $\vec{b} = 3\hat{i} - 4\hat{j} - 2\hat{k}$

   (ii) $\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$ and $\vec{b} = -2\hat{j} + 4\hat{k}$

   (iii) $\vec{a} = \hat{i} - \hat{j} + 5\hat{k}$ and $\vec{b} = 3\hat{i} - 2\hat{k}$

2. Find the value of λ for which $\vec{a}$ and $\vec{b}$ are perpendicular, where

   (i) $\vec{a} = 2\hat{i} + \lambda\hat{j} + \hat{k}$ and $\vec{b} = (\hat{i} - 2\hat{j} + 3\hat{k})$ **[CBSE 2012C]**

   (ii) $\vec{a} = 3\hat{i} - \hat{j} + 4\hat{k}$ and $\vec{b} = -\lambda\hat{i} + 3\hat{j} + 3\hat{k}$

   (iii) $\vec{a} = 2\hat{i} + 4\hat{j} - \hat{k}$ and $\vec{b} = 3\hat{i} - 2\hat{j} + \lambda\hat{k}$ **[CBSE 2003C]**

   (iv) $\vec{a} = 3\hat{i} + 2\hat{j} - 5\hat{k}$ and $\vec{b} = -5\hat{j} + \lambda\hat{k}$

3. (i) If $\vec{a} = \hat{i} + 2\hat{j} - 3\hat{k}$ and $\vec{b} = 3\hat{i} - \hat{j} + 2\hat{k}$, show that $(\vec{a} + \vec{b})$ is perpendicular to $(\vec{a} - \vec{b})$. **[CBSE 2002]**

   (ii) If $\vec{a} = (5\hat{i} - \hat{j} - 3\hat{k})$ and $\vec{b} = (\hat{i} + 3\hat{j} - 5\hat{k})$ then show that $(\vec{a} + \vec{b})$ and $(\vec{a} - \vec{b})$ are orthogonal. **[CBSE 2004]**

4. If $\vec{a} = (\hat{i} - \hat{j} + 7\hat{k})$ and $\vec{b} = (5\hat{i} - \hat{j} + \lambda\hat{k})$ then find the value of λ so that $(\vec{a} + \vec{b})$ and $(\vec{a} - \vec{b})$ are orthogonal vectors. **[CBSE 2013]**

5. Show that the vectors

$$\frac{1}{7}(2\hat{i} + 3\hat{j} + 6\hat{k}), \frac{1}{7}(3\hat{i} - 6\hat{j} + 2\hat{k}) \text{ and } \frac{1}{7}(6\hat{i} + 2\hat{j} - 3\hat{k})$$

are mutually perpendicular unit vectors.

6. Let $\vec{a} = 4\hat{i} + 5\hat{j} - \hat{k}$, $\vec{b} = \hat{i} - 4\hat{j} + 5\hat{k}$ and $\vec{c} = 3\hat{i} + \hat{j} - \hat{k}$.

   Find a vector $\vec{d}$ which is perpendicular to both $\vec{a}$ and $\vec{b}$, and is such that $\vec{d} \cdot \vec{c} = 21$.

7. Let $\vec{a} = (2\hat{i} + 3\hat{j} + 2\hat{k})$ and $\vec{b} = (\hat{i} + 2\hat{j} + \hat{k})$.

   Find the projection of (i) $\vec{a}$ on $\vec{b}$ and (ii) $\vec{b}$ on $\vec{a}$.

8. Find the projection of $(8\hat{i} + \hat{j})$ in the direction of $(\hat{i} + 2\hat{j} - 2\hat{k})$.

9. Write the projection of vector $(\hat{i} + \hat{j} + \hat{k})$ along the vector $\hat{j}$. **[CBSE 2014]**

10. (i) Find the projection of $\vec{a}$ on $\vec{b}$ if $\vec{a} \cdot \vec{b} = 8$ and $\vec{b} = (2\hat{i} + 6\hat{j} + 3\hat{k})$.

    [CBSE 2009]

    (ii) Write the projection of the vector $(\hat{i} + \hat{j})$ on the vector $(\hat{i} - \hat{j})$.

    [CBSE 2011]

11. Find the angle between the vectors $\vec{a}$ and $\vec{b}$, when

    (i) $\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}$ and $\vec{b} = 3\hat{i} - 2\hat{j} + \hat{k}$

    (ii) $\vec{a} = 3\hat{i} + \hat{j} + 2\hat{k}$ and $\vec{b} = 2\hat{i} - 2\hat{j} + 4\hat{k}$

    (iii) $\vec{a} = \hat{i} - \hat{j}$ and $\vec{b} = \hat{j} + \hat{k}$.

12. If $\vec{a} = (\hat{i} + 2\hat{j} - 3\hat{k})$ and $\vec{b} = (3\hat{i} - \hat{j} + 2\hat{k})$ then calculate the angle between $(2\vec{a} + \vec{b})$ and $(\vec{a} + 2\vec{b})$.

13. If $\vec{a}$ is a unit vector such that $(\vec{x} - \vec{a}) \cdot (\vec{x} + \vec{a}) = 8$, find $|\vec{x}|$.

14. Find the angles which the vector $\vec{a} = 3\hat{i} - 6\hat{j} + 2\hat{k}$ makes with the coordinate axes.

15. Show that the vector $\vec{a} = (\hat{i} + \hat{j} + \hat{k})$ is equally inclined to the coordinate axes.

16. Find a vector $\vec{a}$ of magnitude $5\sqrt{2}$, making an angle $\pi/4$ with x-axis, $\pi/2$ with y-axis and an acute angle 0 with z-axis. **[CBSE 2014]**

17. Find the angle between $(\vec{a} + \vec{b})$ and $(\vec{a} - \vec{b})$, if $\vec{a} = (2\hat{i} - \hat{j} + 3\hat{k})$ and $\vec{b} = (3\hat{i} + \hat{j} + 2\hat{k})$. **[CBSE 2006]**

18. Express the vector $\vec{a} = (6\hat{i} - 3\hat{j} - 6\hat{k})$ as sum of two vectors such that one is parallel to the vector $\vec{b} = (\hat{i} + \hat{j} + \hat{k})$ and the other is perpendicular to $\vec{b}$.

19. Prove that $(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b}) = |\vec{a}|^2+|\vec{b}|^2 \Leftrightarrow \vec{a}\perp\vec{b}$, where $\vec{a}\neq\vec{0}$ and $\vec{b}\neq\vec{0}$.

20. If $\vec{a}+\vec{b}+\vec{c}=\vec{0}, |\vec{a}|=3, |\vec{b}|=5$ and $|\vec{c}|=7$, find the angle between $\vec{a}$ and $\vec{b}$. **[CBSE 2008]**

21. Find the angle between $\vec{a}$ and $\vec{b}$, when

(i) $|\vec{a}|=2, |\vec{b}|=1$ and $\vec{a}\cdot\vec{b}=\sqrt{3}$ (ii) $|\vec{a}|=|\vec{b}|=\sqrt{2}$ and $\vec{a}\cdot\vec{b}=-1$.

22. If $|\vec{a}|=2, |\vec{b}|=3$ and $\vec{a}\cdot\vec{b}=4$, find $|\vec{a}-\vec{b}|$.

23. If $(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b})=8$ and $|\vec{a}|=8|\vec{b}|$, find $|\vec{a}|$ and $|\vec{b}|$.

24. If $\hat{a}$ and $\hat{b}$ are unit vectors inclined at an angle θ then prove that:

(i) $\cos\frac{\theta}{2}=\frac{1}{2}|\hat{a}+\hat{b}|$ (ii) $\tan\frac{\theta}{2}=\frac{|\hat{a}-\hat{b}|}{|\hat{a}+\hat{b}|}$

25. The dot products of a vector with the vectors $(\hat{i}+\hat{j}-3\hat{k}), (\hat{i}+3\hat{j}-2\hat{k})$ and $(2\hat{i}+\hat{j}+4\hat{k})$ are 0, 5 and 8 respectively. Find the vector. **[CBSE 2003]**

26. If $\overrightarrow{AB}=(3\hat{i}-\hat{j}+2\hat{k})$ and the coordinates of A are (0, –2, –1), find the coordinates of B.

27. If $A(2, 3, 4), B(5, 4, -1), C(3, 6, 2)$ and $D(1, 2, 0)$ be four points, show that $\overrightarrow{AB}$ is perpendicular to $\overrightarrow{CD}$.

28. Find the value of λ for which the vectors $(2\hat{i}+\lambda\hat{j}+3\hat{k})$ and $(3\hat{i}+2\hat{j}-4\hat{k})$ are perpendicular to each other. **[CBSE 2010]**

29. Show that the vectors $\vec{a}=(3\hat{i}-2\hat{j}+\hat{k}), \vec{b}=(\hat{i}-3\hat{j}+5\hat{k})$ and $\vec{c}=(2\hat{i}+\hat{j}-4\hat{k})$ form a right-angled triangle. **[CBSE 2005]**

30. Three vertices of a triangle are $A(0, -1, -2), B(3, 1, 4)$ and $C(5, 7, 1)$. Show that it is a right-angled triangle. Also, find its other two angles.

31. If the position vectors of the vertices A, B and C of a $\triangle ABC$ be (1, 2, 3), (–1, 0, 0) and (0, 1, 2) respectively then find $\angle ABC$.

32. If $\vec{a}$ and $\vec{b}$ are two unit vectors such that $|\vec{a}+\vec{b}|=\sqrt{3}$, find $(2\vec{a}-5\vec{b})\cdot(3\vec{a}+\vec{b})$.

33. If $\vec{a}$ and $\vec{b}$ are two vectors such that $|\vec{a}+\vec{b}|=|\vec{a}|$ then prove that vector $(2\vec{a}+\vec{b})$ is perpendicular to vector $\vec{b}$. **[CBSE 2013]**

34. If $\vec{a}=(3\hat{i}-\hat{j})$ and $\vec{b}=(2\hat{i}+\hat{j}-3\hat{k})$ then express $\vec{b}$ in the form $\vec{b}=(\vec{b}_1+\vec{b}_2)$, where $\vec{b}_1 \| \vec{a}$ and $\vec{b}_2 \perp \vec{a}$. **[CBSE 2013C]**

ANSWERS (EXERCISE 23)

1. (i) 9 (ii) 8 (iii) –7 **2.** (i) $\lambda=\frac{5}{2}$ (ii) $\lambda=3$ (iii) $\lambda=-2$ (iv) $\lambda=-2$

4. $\lambda=\pm 5$ **6.** $d=7(\hat{i}-\hat{j}-\hat{k})$ **7.** (i) $\frac{5\sqrt{6}}{3}$ (ii) $\frac{10\sqrt{17}}{17}$

8. $\frac{10}{3}$ **9.** 1 **10.** (i) $\frac{8}{7}$ (ii) 0 **11.** (i) $\cos^{-1}\left(\frac{5}{7}\right)$ (ii) $\cos^{-1}\left(\sqrt{\frac{3}{7}}\right)$ (iii) 120°

12. $\cos^{-1}\left(\frac{31}{50}\right)$ **13.** $|\vec{x}|=3$ **14.** $\cos^{-1}\left(\frac{3}{7}\right)$, $\cos^{-1}\left(\frac{-6}{7}\right)$, $\cos^{-1}\left(\frac{2}{7}\right)$

16. $5(\hat{i}+\hat{k})$ **17.** $\frac{\pi}{2}$ **18.** $\vec{a}=-(\hat{i}+\hat{j}+\hat{k})+(7\hat{i}-2\hat{j}-5\hat{k})$ **20.** $\frac{\pi}{3}$

21. (i) $\frac{\pi}{6}$ (ii) $\frac{2\pi}{3}$ **22.** $\sqrt{5}$ **23.** $|\vec{a}|=8\sqrt{\frac{8}{63}}$ and $|\vec{b}|=\sqrt{\frac{8}{63}}$

25. $(\hat{i}+2\hat{j}+\hat{k})$ **26.** $B(3,-3,1)$ **28.** $\lambda=3$

30. $\angle A=45°, \angle B=90°, \angle C=45°$ **31.** $\cos^{-1}\left(\frac{10}{\sqrt{102}}\right)$ **32.** $\frac{-11}{2}$

34. $\vec{b}=(\vec{b}_1+\vec{b}_2)$, where $\vec{b}_1=\left(\frac{3}{2}\hat{i}-\frac{1}{2}\hat{j}\right)$ and $\vec{b}_2=\left(\frac{1}{2}\hat{i}+\frac{3}{2}\hat{j}-3\hat{k}\right)$

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 23)

5. Let the given vectors be $\vec{a}, \vec{b}, \vec{c}$ respectively.

Then, show that $|\vec{a}|=|\vec{b}|=|\vec{c}|=1$, and $\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{c}=\vec{c}\cdot\vec{a}=0$.

6. Let $\vec{d}=d_1\hat{i}+d_2\hat{j}+d_3\hat{k}$. Then,

$\vec{d}\cdot\vec{a}=0$, $\vec{d}\cdot\vec{b}=0$ and $\vec{d}\cdot\vec{c}=21$

$\Rightarrow \begin{Bmatrix} 4d_1+5d_2-d_3=0 \\ d_1-4d_2+5d_3=0 \end{Bmatrix}$ and $3d_1+d_2-d_3=21$

$\Rightarrow \frac{d_1}{(25-4)}=\frac{d_2}{(-1-20)}=\frac{d_3}{(-16-5)}=k$ (say) and $3d_1+d_2-d_3=21$

$\Rightarrow \quad (d_1 = 21k, d_2 = -21k, d_3 = -21k)$ and $3d_1 + d_2 - d_3 = 21$

$\Rightarrow \quad 63k - 21k + 21k = 21 \Rightarrow k = \frac{1}{3} \Rightarrow d_1 = 7, d_2 = -7$ and $d_3 = -7$.

11. (iii) Let the required angle be θ. Then,

$$\vec{a} \cdot \vec{b} = (1 \times 0) + (-1) \times 1 + (0 \times 1) = -1$$

$$\Rightarrow |\vec{a}||\vec{b}| \cos\theta = -1$$

$$\Rightarrow (\sqrt{2})(\sqrt{2}) \cos\theta = -1 \Rightarrow \cos\theta = -\frac{1}{2} \Rightarrow \theta = 120°.$$

12. $(2\vec{a} + \vec{b}) \cdot (\vec{a} + 2\vec{b}) = |2\vec{a} + \vec{b}| \cdot |\vec{a} + 2\vec{b}| \cos\theta$. Find θ.

13. $(\vec{x} - \vec{a}) \cdot (\vec{x} + \vec{a}) = 8 \Rightarrow \vec{x} \cdot \vec{x} - \vec{a} \cdot \vec{a} = 8 \quad [\because \quad \vec{x} \cdot \vec{a} = \vec{a} \cdot \vec{x}]$

$$\Rightarrow |\vec{x}|^2 - |\vec{a}|^2 = 8 \Rightarrow |\vec{x}|^2 = (8 + 1) = 9 \Rightarrow |x| = 3.$$

14. $\vec{a} \cdot \hat{i} = |\vec{a}||\hat{i}| \cos\alpha \Rightarrow 7 \times 1 \times \cos\alpha = 3 \Rightarrow \cos\alpha = \frac{3}{7}$.

$\vec{a} \cdot \hat{j} = |\vec{a}||\hat{j}| \cos\beta \Rightarrow 7 \times 1 \times \cos\beta = -6 \Rightarrow \cos\beta = \frac{-6}{7}$.

$\vec{a} \cdot \hat{k} = |\vec{a}||\hat{k}| \cos\gamma \Rightarrow 7 \times 1 \times \cos\gamma = 2 \Rightarrow \cos\gamma = \frac{2}{7}$.

15. $\vec{a} \cdot \hat{i} = |\vec{a}||\hat{i}| \cos\alpha \Rightarrow \sqrt{3} \times 1 \times \cos\alpha = 1 \Rightarrow \cos\alpha = \frac{1}{\sqrt{3}}$.

Similarly, $\cos\beta = \frac{1}{\sqrt{3}}$ and $\cos\gamma = \frac{1}{\sqrt{3}}$.

$\therefore \quad \alpha = \beta = \gamma = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$.

16. Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$. Then, $|\vec{a}|^2 = 50 \Rightarrow (a_1^2 + a_2^2 + a_3^2) = 50$.

$\vec{a} \cdot \hat{i} = |\vec{a}||\hat{i}| \cos\alpha \Rightarrow a_1 = \sqrt{50} \times 1 \times \cos\frac{\pi}{4} = 5\sqrt{2} \times \frac{1}{\sqrt{2}} = 5$.

$\vec{a} \cdot \hat{j} = |\vec{a}||\hat{j}| \cos\beta \Rightarrow a_2 = \sqrt{50} \times 1 \times \cos\frac{\pi}{2} = 0$

$\vec{a} \cdot \hat{k} = |\vec{a}||\hat{k}| \cos\gamma \Rightarrow a_3 = \sqrt{50} \times 1 \times \cos\theta$

$(a_1^2 + a_2^2 + a_3^2) = 50 \Rightarrow 25 + 0 + 50\cos^2\theta = 50$

$$\Rightarrow \cos^2\theta = \frac{1}{2} \Rightarrow \cos\theta = \frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{\pi}{2}.$$

$\therefore \quad a_3 = \sqrt{50} \times \cos\frac{\pi}{4} = \left(5\sqrt{2} \times \frac{1}{\sqrt{2}}\right) = 5.$

$\therefore \quad \vec{a} = 5(\hat{i} + \hat{k})$.

17. Let $\vec{u} = (\vec{a} + \vec{b}) = (5\hat{i} + \hat{k})$ and $\vec{v} = (\vec{a} - \vec{b}) = (-\hat{i} - 2\hat{j} + 5\hat{k})$.

Then, $|\vec{u}| = \sqrt{5^2 + 1^2} = \sqrt{26}$ and $|\vec{v}| = \sqrt{(-1)^2 + (-2)^2 + 5^2} = \sqrt{30}.$

$$\cos\theta = \frac{(\vec{u}\cdot\vec{v})}{|\vec{u}||\vec{v}|} = \frac{(-5-0+5)}{\sqrt{26}\cdot\sqrt{30}} = 0 \Rightarrow \theta = \frac{\pi}{2}.$$

18. Let $\vec{a} = \lambda\vec{b} + \vec{c}$, where $\vec{c} \perp \vec{b}$. Then, $\vec{c} = (\vec{a} - \lambda\vec{b}) \perp \vec{b}$.

$$\therefore \quad (\vec{a} - \lambda\vec{b})\cdot\vec{b} = 0 \Rightarrow (\vec{a}\cdot\vec{b}) - \lambda(\vec{b}\cdot\vec{b}) = 0$$

$$\Rightarrow (6 - 3 - 6) - \lambda(1 + 1 + 1) = 0 \Rightarrow \lambda = \frac{-3}{3} = -1.$$

$$\therefore \quad \vec{a} = -\vec{b} + \vec{c} \Rightarrow \vec{c} = (\vec{a} + \vec{b}).$$

20. $\vec{a} + \vec{b} = -\vec{c} \Rightarrow (\vec{a} + \vec{b})\cdot(\vec{a} + \vec{b}) = (-\vec{c})\cdot(-\vec{c})$

$$\Rightarrow |\vec{a}|^2 + |\vec{b}|^2 + 2\vec{a}\cdot\vec{b} = |\vec{c}|^2 \Rightarrow \vec{a}\cdot\vec{b} = \frac{15}{2}$$

$$\Rightarrow ab\cos\theta = \frac{15}{2} \Rightarrow (3\times 5)\cos\theta = \frac{15}{2} \Rightarrow \cos\theta = \frac{1}{2}.$$

21. $$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}.$$

22. $|\vec{a} - \vec{b}|^2 = (\vec{a} - \vec{b})\cdot(\vec{a} - \vec{b})$

$$= \vec{a}\cdot\vec{a} - \vec{a}\cdot\vec{b} - \vec{b}\cdot\vec{a} + \vec{b}\cdot\vec{b}$$

$$= |\vec{a}|^2 - 2(\vec{a}\cdot\vec{b}) + |\vec{b}|^2 = 5.$$

23. $|\vec{a}|^2 - |\vec{b}|^2 = 8 \Rightarrow 64|\vec{b}|^2 - |\vec{b}|^2 = 8 \Rightarrow |\vec{b}|^2 = \frac{8}{63}.$

$$\therefore \quad |\vec{b}| = \sqrt{\frac{8}{63}} \text{ and } |\vec{a}| = 8\cdot\sqrt{\frac{8}{63}}.$$

24. $|\hat{a} + \hat{b}|^2 = (\hat{a} + \hat{b})\cdot(\hat{a} + \hat{b}) = \hat{a}\cdot\hat{a} + \hat{a}\cdot\hat{b} + \hat{b}\cdot\hat{a} + \hat{b}\cdot\hat{b}$

$$= |\hat{a}|^2 + 2(\hat{a}\cdot\hat{b}) + |\hat{b}|^2 = 1 + 2|\hat{a}||\hat{b}|\cos\theta + 1$$

$$= 2(1 + \cos\theta) = 2\times 2\cos^2\frac{\theta}{2} = 4\cos^2\frac{\theta}{2}.$$

$$\therefore \quad \cos\frac{\theta}{2} = \frac{1}{2}|\hat{a} + \hat{b}|.$$

Similarly, we have $|\hat{a} - \hat{b}|^2 = 4\sin^2\frac{\theta}{2}.$

$$\therefore \quad \sin\frac{\theta}{2} = \frac{1}{2}|\hat{a} - \hat{b}|.$$

Hence, $$\tan\frac{\theta}{2} = \frac{|\hat{a} - \hat{b}|}{|\hat{a} + \hat{b}|}.$$

25. Let the given vectors be $\vec{a}$, $\vec{b}$ and $\vec{c}$ respectively and let the required vector be $\vec{d} = d_1\hat{i} + d_2\hat{j} + d_3\hat{k}$.

Then, $\vec{d}\cdot\vec{a} = 0$, $\vec{d}\cdot\vec{b} = 5$ and $\vec{d}\cdot\vec{c} = 8$

$\Rightarrow \quad d_1 + d_2 - 3d_3 = 0$ ... (i), $d_1 + 3d_2 - 2d_3 = 5$ ... (ii) and $2d_1 + d_2 + 4d_3 = 8$. ... (iii)

On subtracting (i) from (ii), we get $2d_2 + d_3 = 5$. ... (iv)

On multiplying (i) by 2 and subtracting (iii) from it, we get $d_2 - 10d_3 = -8$. ... (v)

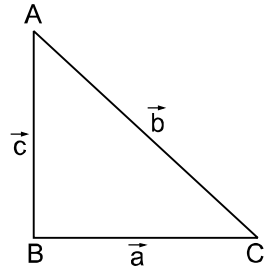
On solving (iv) and (v), we get $d_2 = 2$, $d_3 = 1$ and therefore, $d_1 = 1$.

26. $\overrightarrow{AB} = (\overrightarrow{OB} - \overrightarrow{OA}) \Rightarrow \overrightarrow{OB} = (\overrightarrow{AB} + \overrightarrow{OA})$.

27. $\overrightarrow{AB} = (\overrightarrow{OB} - \overrightarrow{OA}) = (3\hat{i} + \hat{j} - 5\hat{k})$, $\overrightarrow{CD} = (\overrightarrow{OD} - \overrightarrow{OC}) = (-2\hat{i} + 4\hat{j} - 2\hat{k})$.

29. Clearly, $\vec{b} + \vec{c} = \vec{a}$ and $\vec{a}\cdot\vec{c} = 0$.

$\therefore \quad \triangle ABC$ is right angled at B.



30. Let O be the origin. Then,

$\overrightarrow{OA} = (-\hat{j} - 2\hat{k})$, $\overrightarrow{OB} = (3\hat{i} + \hat{j} + 4\hat{k})$ and $\overrightarrow{OC} = (5\hat{i} + 7\hat{j} + \hat{k})$.

$\therefore \quad \overrightarrow{AB} = (\overrightarrow{OB} - \overrightarrow{OA}) = (3\hat{i} + 2\hat{j} + 6\hat{k})$, $\overrightarrow{BC} = (\overrightarrow{OC} - \overrightarrow{OB}) = (2\hat{i} + 6\hat{j} - 3\hat{k})$ and

$\overrightarrow{CA} = (-5\hat{i} - 8\hat{j} - 3\hat{k})$.

Now, $\overrightarrow{AB}\cdot\overrightarrow{BC} = (3\hat{i} + 2\hat{j} + 6\hat{k})\cdot(2\hat{i} + 6\hat{j} - 3\hat{k}) = (6 + 12 - 18) = 0$.

$\therefore \quad \overrightarrow{AB} \perp \overrightarrow{BC}$ and therefore, $\angle B = 90°$.

Hence, $\triangle ABC$ is right angled at B.

Now, $\overrightarrow{AC} = -\overrightarrow{CA} = (5\hat{i} + 8\hat{j} + 3\hat{k})$.

$$\cos A = \frac{\overrightarrow{AB}\cdot\overrightarrow{AC}}{|\overrightarrow{AB}||\overrightarrow{AC}|} = \frac{(3\times5 + 2\times8 + 6\times3)}{\left\{\sqrt{3^2 + 2^2 + 6^2}\right\}\left\{\sqrt{5^2 + 8^2 + 3^2}\right\}}$$

$$= \frac{49}{(\sqrt{49})(\sqrt{98})} = \frac{1}{\sqrt{2}}$$

$\Rightarrow \quad \angle A = 45°$ and therefore, $\angle C = 180° - (90° + 45°) = 45°$.

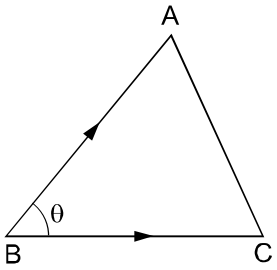
31. Let $\angle ABC = \theta$.

Now, $\overrightarrow{BA} = (\text{p.v. of } A) - (\text{p.v.of } B) = (2\hat{i} + 2\hat{j} + 3\hat{k})$

and $\overrightarrow{BC} = (\text{p.v. of } C) - (\text{p.v.of } B) = (\hat{i} + \hat{j} + 2\hat{k})$.

$$\cos\theta = \frac{\overrightarrow{BA}\cdot\overrightarrow{BC}}{|\overrightarrow{BA}||\overrightarrow{BC}|} = \frac{(2\times1+2\times1+3\times2)}{\left\{\sqrt{2^2+2^2+3^2}\right\}\left\{\sqrt{1^2+1^2+2^2}\right\}}$$

$$= \frac{10}{\sqrt{17}\times\sqrt{6}} = \frac{10}{\sqrt{102}}$$



$$\Rightarrow \quad \theta = \cos^{-1}\left\{\frac{10}{\sqrt{102}}\right\}.$$

32. Given: $|\vec{a}| = 1, |\vec{b}| = 1$ and $|\vec{a}+\vec{b}| = \sqrt{3}$.

$$\therefore \quad |\vec{a}+\vec{b}|^2 = (\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b})$$

$$= (\vec{a}\cdot\vec{a}) + 2(\vec{a}\cdot\vec{b}) + (\vec{b}\cdot\vec{b})$$

$$= |\vec{a}|^2 + 2(\vec{a}\cdot\vec{b}) + |\vec{b}|^2 = 2 + 2(\vec{a}\cdot\vec{b}).$$

$$\therefore \quad 2[1+(\vec{a}\cdot\vec{b})] = 3 \quad \Rightarrow \quad (\vec{a}\cdot\vec{b}) = \frac{1}{2}.$$

Now, find $(2\vec{a}-5\vec{b})\cdot(3\vec{a}+\vec{b})$.

33. $|\vec{a}+\vec{b}| = |\vec{a}| \Rightarrow |\vec{a}+\vec{b}|^2 = |\vec{a}|^2$

$$\Rightarrow (\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b}) = |\vec{a}|^2$$

$$\Rightarrow \vec{a}\cdot\vec{a} + 2(\vec{a}\cdot\vec{b}) + \vec{b}\cdot\vec{b} = |\vec{a}|^2 \quad [\because \ (\vec{b}\cdot\vec{a}) = (\vec{a}\cdot\vec{b})]$$

$$\Rightarrow 2(\vec{a}\cdot\vec{b}) + \vec{b}\cdot\vec{b} = 0 \Rightarrow 2\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{b} = 0$$

$$\Rightarrow (2\vec{a}+\vec{b})\cdot\vec{b} = 0.$$

34. Let $\vec{b} = \vec{b}_1 + \vec{b}_2$, where $\vec{b}_1 \| \vec{a}$ and $\vec{b}_2 \perp \vec{a}$.

Let $\vec{b}_1 = \lambda(3\hat{i}-\hat{j})$ and let $\vec{b}_2 = x\hat{i}+y\hat{j}+z\hat{k}$. Then,

$$\vec{b}_2 \perp \vec{a} \Rightarrow (x\hat{i}+y\hat{j}+z\hat{k})\cdot(3\hat{i}-\hat{j}) = 0 \Rightarrow 3x - y = 0.$$

$$\therefore \quad \vec{b} = (\vec{b}_1 + \vec{b}_2) \Rightarrow (2\hat{i}+\hat{j}-3\hat{k}) = \lambda(3\hat{i}-\hat{j}) + (x\hat{i}+y\hat{j}+z\hat{k})$$

$$\Rightarrow (2\hat{i}+\hat{j}-3\hat{k}) = (3\lambda+x)\hat{i} + (y-\lambda)\hat{j} + z\hat{k}$$

$$\Rightarrow z = -3,\ y-\lambda = 1 \text{ and } 3\lambda + x = 2$$

$$\Rightarrow z = -3,\ \lambda = (y-1) \text{ and } 3(y-1)+x = 2$$

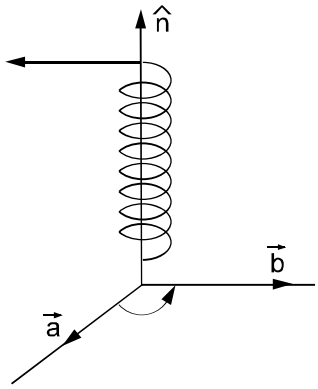
$$\Rightarrow z = -3,\ \lambda = (y-1) \text{ and } 3y + x = 5.$$

On solving $3x - y = 0$ and $x + 3y = 5$, we get $x = \frac{1}{2}$ and $y = \frac{3}{2}, z = -3$ and $\lambda = \frac{1}{2}$.

$$\therefore \quad \vec{b}_1 = \frac{1}{2}(3\hat{i}-\hat{j}) \text{ and } \vec{b}_2 = \frac{1}{2}\hat{i} + \frac{3}{2}\hat{j} - 3\hat{k}.$$

24. CROSS, OR VECTOR, PRODUCT OF VECTORS

VECTOR PRODUCT OF TWO VECTORS Let $\vec{a}$ and $\vec{b}$ be two nonzero, nonparallel vectors, and let θ be the angle between them such that $0 < \theta < \pi$.



Then, the vector product of $\vec{a}$ and $\vec{b}$ is defined as

$$\vec{a} \times \vec{b} = (|\vec{a}||\vec{b}| \sin\theta)\,\hat{n},$$

where $\hat{n}$ is a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$, such that $\vec{a}, \vec{b}, \hat{n}$ form a right-handed system.

When a right-handed screw is rotated from $\vec{a}$ to $\vec{b}$ and it advances along $\hat{n}$ then the system is said to be right handed.

REMARK 1 If $\vec{a}$ and $\vec{b}$ are parallel or collinear, i.e., when $\theta = 0$ or $\theta = \pi$ then we define, $\vec{a} \times \vec{b} = \vec{0}$.

In particular, $\vec{a} \times \vec{a} = \vec{0}$.

REMARK 2 If $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$, we define, $\vec{a} \times \vec{b} = \vec{0}$.

REMARK 3 For any vector $\vec{a}$, we have $\vec{a} \times \vec{a} = (|\vec{a}||\vec{a}| \sin 0)\,\hat{n} = 0\,\hat{n} = \vec{0}$.

ANGLE BETWEEN TWO VECTORS Let θ be the angle between $\vec{a}$ and $\vec{b}$. Then,

$$\vec{a} \times \vec{b} = (a\,b \sin\theta)\,\hat{n}, \text{ where } |\vec{a}| = a \text{ and } |\vec{b}| = b$$

$$\Rightarrow \quad |\vec{a} \times \vec{b}| = a\,b \sin\theta$$

$$\Rightarrow \quad \sin\theta = \frac{|\vec{a} \times \vec{b}|}{ab} = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}||\vec{b}|}$$

$$\Rightarrow \quad \theta = \sin^{-1}\left\{\frac{|\vec{a} \times \vec{b}|}{|\vec{a}||\vec{b}|}\right\}.$$

UNIT VECTOR PERPENDICULAR TO TWO GIVEN VECTORS A unit vector $\hat{n}$ perpendicular to each one of $\vec{a}$ and $\vec{b}$ is given by

$$\hat{n} = \frac{(\vec{a} \times \vec{b})}{|\vec{a} \times \vec{b}|}.$$

Properties of Vector Product

RESULT 1 *Vector product is not commutative.*

In fact, we have $(\vec{b} \times \vec{a}) = -(\vec{a} \times \vec{b})$.

PROOF Let $\vec{a}, \vec{b}, \hat{n}$ form a right-handed system. Then,

$$(\vec{a} \times \vec{b}) = (a\, b \sin\theta)\, \hat{n}. \qquad \text{... (i)}$$

Then, $\vec{b}, \vec{a}, -\hat{n}$ form a right-handed system.

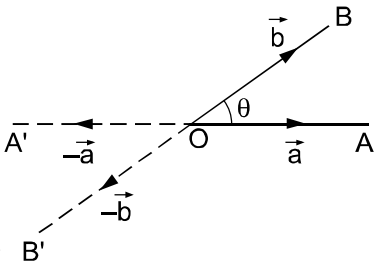
$$\therefore \ (\vec{b} \times \vec{a}) = (b\, a \sin 0)\, (-\hat{n})$$

$$\Rightarrow \ -(\vec{b} \times \vec{a}) = (a\, b \sin\theta)\, \hat{n}. \qquad \text{... (ii)}$$

From (i) and (ii), we get $(\vec{a} \times \vec{b}) = -(\vec{b} \times \vec{a})$.

RESULT 2 *For any two vectors $\vec{a}$ and $\vec{b}$, prove that*

$$(-\vec{b}) \times \vec{a} = (\vec{a} \times \vec{b}) = \vec{b} \times (-\vec{a}).$$



PROOF Let θ be the angle between $\vec{a}$ and $\vec{b}$.

Then, the angle between $(-\vec{b})$ and $\vec{a}$ is $(\pi - \theta)$.

And, $(-\vec{b}), \vec{a}, \hat{n}$ form a right-handed system.

$$\therefore \quad (-\vec{b}) \times \vec{a} = |-\vec{b}|\,|\vec{a}| \sin(\pi - \theta)\, \hat{n}$$

$$= (|\vec{b}|\,|\vec{a}| \sin\theta)\, \hat{n}$$

$$= (a\, b \sin\theta)\, \hat{n} = (\vec{a} \times \vec{b}).$$

$$\therefore \quad (-\vec{b}) \times \vec{a} = (\vec{a} \times \vec{b}).$$

Similarly, $\vec{b} \times (-\vec{a}) = (\vec{a} \times \vec{b})$.

Hence, $(-\vec{b}) \times \vec{a} = (\vec{a} \times \vec{b}) = \vec{b} \times (-\vec{a})$.

RESULT 3 *For any scalar m, we have $(m\vec{a} \times \vec{b}) = m(\vec{a} \times \vec{b}) = \vec{a} \times (m\vec{b})$.*

AN IMPORTANT NOTE At a later stage, we shall prove that for any three vectors $\vec{a}, \vec{b}, \vec{c}$, we have $\vec{a} \times \vec{b} \cdot \vec{c} = \vec{a} \cdot \vec{b} \times \vec{c}$, i.e., the position of dot and cross can be interchanged. However, we shall use this fact in proving the following theorem.

RESULT 4 *(Distributive law) For any vectors $\vec{a}, \vec{b}, \vec{c}$, prove that*

$$\vec{a} \times (\vec{b} + \vec{c}) = (\vec{a} \times \vec{b}) + (\vec{a} \times \vec{c}).$$

PROOF Let $\vec{p} = [\vec{a} \times (\vec{b} + \vec{c}) - (\vec{a} \times \vec{b}) - (\vec{a} \times \vec{c})]$, and let $\vec{r}$ be any arbitrary vector.

Then,

$$\begin{aligned}\vec{r}\cdot\vec{p} &= \vec{r}\cdot[\vec{a}\times(\vec{b}+\vec{c})-(\vec{a}\times\vec{b})-(\vec{a}\times\vec{c})]\\ &= \vec{r}\cdot[\vec{a}\times(\vec{b}+\vec{c})]-\vec{r}\cdot(\vec{a}\times\vec{b})-\vec{r}\cdot(\vec{a}\times\vec{c})\\ &= (\vec{r}\times\vec{a})\cdot(\vec{b}+\vec{c})-(\vec{r}\times\vec{a})\cdot\vec{b}-(\vec{r}\times\vec{a})\cdot\vec{c}\\ &= (\vec{r}\times\vec{a})\cdot\vec{b}+(\vec{r}\times\vec{a})\cdot\vec{c}-(\vec{r}\times\vec{a})\cdot\vec{b}-(\vec{r}\times\vec{a})\cdot\vec{c}\\ &= 0.\end{aligned}$$

Now, $\vec{r}\cdot\vec{p}=0 \Rightarrow \vec{r}=\vec{0}$ or $\vec{p}=\vec{0}$ or $(\vec{r}\perp\vec{p})$.

Since $\vec{r}$ is an arbitrary vector, we may choose it in such a way that $\vec{r}\neq\vec{0}$ and $\vec{r}$ is not perpendicular to $\vec{p}$.

Then, $\vec{p}=\vec{0}$

$$\Rightarrow \vec{a}\times(\vec{b}+\vec{c})-(\vec{a}\times\vec{b})-(\vec{a}\times\vec{c})=\vec{0}$$

$$\Rightarrow \vec{a}\times(\vec{b}+\vec{c})=(\vec{a}\times\vec{b})+(\vec{a}\times\vec{c}).$$

Hence, $\vec{a}\times(\vec{b}+\vec{c})=(\vec{a}\times\vec{b})+(\vec{a}\times\vec{c})$.

RESULT 5 *For any three vectors $\vec{a}$, $\vec{b}$, $\vec{c}$, prove that*

$$\vec{a}\times(\vec{b}-\vec{c})=(\vec{a}\times\vec{b})-(\vec{a}\times\vec{c}).$$

PROOF We have

$$\begin{aligned}\vec{a}\times(\vec{b}-\vec{c}) &= \vec{a}\times[\vec{b}+(-\vec{c})]\\ &= (\vec{a}\times\vec{b})+\vec{a}\times(-\vec{c}) \quad \text{[by the distributive law]}\\ &= (\vec{a}\times\vec{b})+[-(\vec{a}\times\vec{c})] \quad [\because\ \vec{a}\times(-\vec{c})=-(\vec{a}\times\vec{c})]\\ &= (\vec{a}\times\vec{b})-(\vec{a}\times\vec{c}).\end{aligned}$$

Hence, $\vec{a}\times(\vec{b}-\vec{c})=(\vec{a}\times\vec{b})-(\vec{a}\times\vec{c})$.

RESULT 6 *Prove that two nonzero vectors $\vec{a}$ and $\vec{b}$ are parallel or collinear if and only if $(\vec{a}\times\vec{b})=\vec{0}$.*

PROOF Let $\vec{a}\neq\vec{0}$ and $\vec{b}\neq\vec{0}$ and let θ be the angle between them.

Let $\vec{a}$ and $\vec{b}$ be parallel or collinear.

Then, $\theta=0$ or $\theta=\pi$

$$\Rightarrow \quad \sin\theta=0 \quad \Rightarrow \quad (\vec{a}\times\vec{b})=(ab\sin\theta)\,\hat{n}=0\,\hat{n}=\vec{0}.$$

Thus, when $\vec{a}$ and $\vec{b}$ are parallel or collinear, then $(\vec{a}\times\vec{b})=\vec{0}$.

Again, let $\vec{a} \neq \vec{0}$, $\vec{b} \neq \vec{0}$ and $(\vec{a} \times \vec{b}) = \vec{0}$. Then,

$$(\vec{a} \times \vec{b}) = \vec{0} \Rightarrow |\vec{a} \times \vec{b}| = 0$$
$$\Rightarrow ab \sin\theta = 0$$
$$\Rightarrow \sin\theta = 0 \quad [a \neq 0 \text{ and } b \neq 0]$$
$$\Rightarrow \theta = 0 \text{ or } \theta = \pi$$
$$\Rightarrow (\vec{a} \parallel \vec{b}) \text{ or } (\vec{a} \text{ and } \vec{b} \text{ are collinear}).$$

RESULT 7 *Prove that*

$$(\vec{a} \times \vec{b}) = \vec{0} \Rightarrow \vec{a} = \vec{0} \text{ or } \vec{b} = \vec{0} \text{ or } \vec{a} \parallel \vec{b} \text{ or } \vec{a} \text{ and } \vec{b} \text{ are collinear.}$$

PROOF Let θ be the angle between $\vec{a}$ and $\vec{b}$. Then,

$$\vec{a} \times \vec{b} = \vec{0} \Rightarrow (ab \sin\theta)\hat{n} = \vec{0}$$
$$\Rightarrow ab \sin\theta = 0$$
$$\Rightarrow a = 0 \text{ or } b = 0 \text{ or } \sin\theta = 0$$
$$\Rightarrow \vec{a} = \vec{0} \text{ or } \vec{b} = \vec{0} \text{ or } \theta = 0 \text{ or } \theta = \pi$$
$$\Rightarrow \vec{a} = \vec{0} \text{ or } \vec{b} = \vec{0} \text{ or } \vec{a} \text{ and } \vec{b} \text{ are parallel or collinear.}$$

COROLLARY *Show that* $\vec{a} \times \vec{b} = \vec{a} \times \vec{c}$ *does not imply that* $\vec{b} = \vec{c}$.

PROOF $\vec{a} \times \vec{b} = \vec{a} \times \vec{c} \Rightarrow \vec{a} \times \vec{b} - \vec{a} \times \vec{c} = \vec{0}$

$$\Rightarrow \vec{a} \times (\vec{b} - \vec{c}) = \vec{0}$$
$$\Rightarrow \vec{a} = \vec{0} \text{ or } (\vec{b} - \vec{c}) = 0 \text{ or } \vec{a} \parallel (\vec{b} - \vec{c})$$
$$\Rightarrow \vec{a} = \vec{0} \text{ or } \vec{b} = \vec{c} \text{ or } \vec{a} \parallel (\vec{b} - \vec{c}).$$

$\therefore$ $\vec{a} \times \vec{b} = \vec{a} \times \vec{c}$ does not always mean that $\vec{b} = \vec{c}$.

SUMMARY

(i) $(\vec{a} \times \vec{b}) = -(\vec{b} \times \vec{a})$

(ii) $(-\vec{b}) \times \vec{a} = (\vec{a} \times \vec{b}) = \vec{b} \times (-\vec{a})$

(iii) $m\,\vec{a} \times \vec{b} = m(\vec{a} \times \vec{b}) = \vec{a} \times (m\,\vec{b})$

(iv) $\vec{a} \times (\vec{b} + \vec{c}) = (\vec{a} \times \vec{b}) + (\vec{a} \times \vec{c})$

(v) $\vec{a} \times (\vec{b} - \vec{c}) = (\vec{a} \times \vec{b}) - (\vec{a} \times \vec{c})$

(vi) $\vec{a} \times \vec{b} = \vec{0} \Leftrightarrow (\vec{a} \parallel \vec{b})$ or $(\vec{a}$ and $\vec{b}$ are collinear), when $\vec{a} \neq \vec{0}$ and $\vec{b} \neq \vec{0}$

Area of a Parallelogram

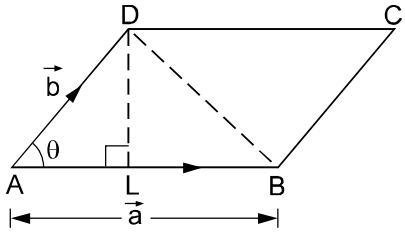
THEOREM 1 *Two adjacent sides of a ||gm are represented by* $\vec{a}$ *and* $\vec{b}$ *respectively. Prove that the area of the ||gm* $= |\vec{a} \times \vec{b}|$.

PROOF Let $ABCD$ be a ||gm. Join BD.

Let $\overrightarrow{AB} = \vec{a}$, $\overrightarrow{AD} = \vec{b}$ and $\angle BAD = \theta$.

Draw $DL \perp AB$. Then,

$AB = a$ and $DL = AD \sin\theta = b \sin\theta$.



$$\therefore \quad \text{ar}(\triangle ABD) = \frac{1}{2} \times \text{base} \times \text{altitude}$$

$$= \left(\frac{1}{2} \times AB \times DL\right) = \frac{1}{2}\, a\, b \sin\theta$$

$$= \frac{1}{2} |\vec{a} \times \vec{b}|.$$

$$\therefore \quad \text{ar}(\text{||gm } ABCD) = 2 \times \text{ar}(\triangle ABD) = |\vec{a} \times \vec{b}|.$$

REMARK $(\vec{a} \times \vec{b})$ is called the vector area of the ||gm.

Area of a Triangle

THEOREM 2 *Prove that the area of* $\triangle ABC$, *where* $\overrightarrow{AB} = \vec{a}$ *and* $\overrightarrow{AC} = \vec{b}$, *is* $\frac{1}{2}|\vec{a} \times \vec{b}|$.

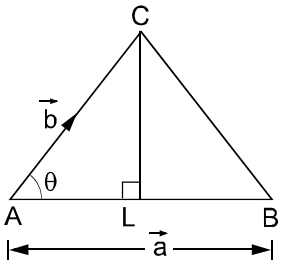
PROOF In $\triangle ABC$, let $\overrightarrow{AB} = \vec{a}$ and $\overrightarrow{AC} = \vec{b}$ and $\angle BAC = \theta$.

Draw $CL \perp AB$. Then,

$AB = a$ and $CL = (AC) \sin\theta = b \sin\theta$

and $\quad \text{ar}(\triangle ABC) = \frac{1}{2} \times AB \times CL$

$$= \frac{1}{2}\, a\, b \sin\theta = \frac{1}{2} |\vec{a} \times \vec{b}|.$$



$$\therefore \quad \text{ar}(\triangle ABC) = \frac{1}{2} |\vec{a} \times \vec{b}|.$$

REMARK $\frac{1}{2}(\vec{a} \times \vec{b})$ is called the vector area of $\triangle ABC$.

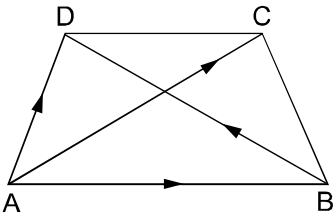
Area of a Quadrilateral

THEOREM 3 *Prove that the area of a quadrilateral ABCD with diagonals AC and BD is* $\frac{1}{2}|\overrightarrow{AC} \times \overrightarrow{BD}|$.

PROOF Vector area of quad. $ABCD$

= (vector area of $\triangle ABC$) + (vector area of $\triangle ACD$)

$$= \frac{1}{2}(\overrightarrow{AB} \times \overrightarrow{AC}) + \frac{1}{2}(\overrightarrow{AC} \times \overrightarrow{AD})$$



$$= -\frac{1}{2}(\overrightarrow{AC} \times \overrightarrow{AB}) + \frac{1}{2}(\overrightarrow{AC} \times \overrightarrow{AD})$$

$$= \frac{1}{2} \cdot \{\overrightarrow{AC} \times (\overrightarrow{AD} - \overrightarrow{AB})\} = \frac{1}{2}(\overrightarrow{AC} \times \overrightarrow{BD}).$$

$$\therefore \quad \text{ar(quad. } ABCD) = \frac{1}{2} | \overrightarrow{AC} \times \overrightarrow{BD} |.$$

SUMMARY

(i) ar (||gm $ABCD$) $= | \vec{a} \times \vec{b} |$, where $\overrightarrow{AB} = \vec{a}$ and $\overrightarrow{AD} = \vec{b}$.

(ii) ar (||gm $ABCD$) $= \frac{1}{2} | \vec{d}_1 \times \vec{d}_2 |$, where $\vec{d}_1$ and $\vec{d}_2$ are the diagonal vectors.

(iii) ar$(\triangle ABC) = \frac{1}{2} | \vec{a} \times \vec{b} |$, where $\overrightarrow{AB} = \vec{a}$ and $\overrightarrow{AC} = \vec{b}$.

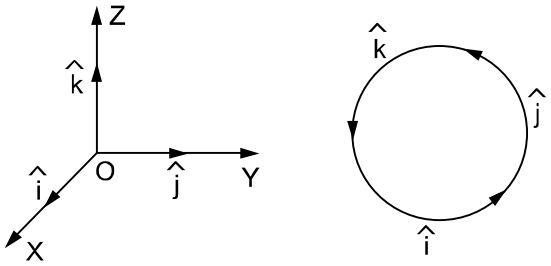
(iv) ar (quad. $ABCD$) $= \frac{1}{2} | \overrightarrow{AB} \times \overrightarrow{BD} |$, where AC and BD are its diagonals.

VECTOR PRODUCT OF AN ORTHONORMAL VECTOR TRIAD

For mutually perpendicular unit vectors $\hat{i}$, $\hat{j}$, $\hat{k}$, we have

$$\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = \hat{0},$$

$$\hat{i} \times \hat{j} = \hat{k} = -\hat{j} \times \hat{i}, \quad \hat{j} \times \hat{k} = \hat{i} = -\hat{k} \times \hat{j} \quad \text{and} \quad \hat{k} \times \hat{i} = \hat{j} = -\hat{i} \times \hat{k}.$$



Vector Product in Terms of Components

Let $\vec{a} = a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}$ *and* $\vec{b} = b_1 \hat{i} + b_2 \hat{j} + b_3 \hat{k}$. *Then,*

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

PROOF We have

$$\vec{a} \times \vec{b} = (a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}) \times (b_1 \hat{i} + b_2 \hat{j} + b_3 \hat{k})$$

$$= a_1 b_1 (\hat{i} \times \hat{i}) + a_1 b_2 (\hat{i} \times \hat{j}) + a_1 b_3 (\hat{i} \times \hat{k})$$

$$+ a_2b_1(\hat{j} \times \hat{i}) + a_2b_2(\hat{j} \times \hat{j}) + a_2b_3(\hat{j} \times \hat{k})$$
$$+ a_3b_1(\hat{k} \times \hat{i}) + a_3b_2(\hat{k} \times \hat{j}) + a_3b_3(\hat{k} \times \hat{k})$$
$$= (a_2b_3 - a_3b_2)\,\hat{i} + (a_3b_1 - a_1b_3)\,\hat{j} + (a_1b_2 - a_2b_1)\,\hat{k}$$

$$\left[\because\ \hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0,\ \hat{i} \times \hat{j} = \hat{k},\ \hat{j} \times \hat{k} = \hat{i},\ \hat{k} \times \hat{i} = \hat{j},\ \text{and } \hat{j} \times \hat{i} = -\hat{k},\ \hat{k} \times \hat{j} = -\hat{i} \text{ and } \hat{i} \times \hat{k} = -\hat{j}\right]$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

SOLVED EXAMPLES

EXAMPLE 1 *If* $\vec{a} = (3\hat{i} + \hat{j} - 4\hat{k})$ *and* $\vec{b} = (6\hat{i} + 5\hat{j} - 2\hat{k})$, *find* $(\vec{a} \times \vec{b})$ *and* $|\vec{a} \times \vec{b}|$.

SOLUTION We have

$$(\vec{a} \times \vec{b}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 1 & -4 \\ 6 & 5 & -2 \end{vmatrix}$$
$$= (-2 + 20)\,\hat{i} - (-6 + 24)\,\hat{j} + (15 - 6)\,\hat{k}$$
$$= (18\hat{i} - 18\hat{j} + 9\hat{k}).$$
$$|\vec{a} \times \vec{b}|^2 = \{(18)^2 + (-18)^2 + 9^2\} = 729$$
$$\Rightarrow \quad |\vec{a} \times \vec{b}| = \sqrt{729} = 27.$$

EXAMPLE 2 *If* $\vec{a} = (\hat{i} - 2\hat{j} + 3\hat{k})$ *and* $\vec{b} = (2\hat{i} + 3\hat{j} - 5\hat{k})$ *then find* $(\vec{a} \times \vec{b})$ *and verify that* $(\vec{a} \times \vec{b})$ *is perpendicular to each one of* $\vec{a}$ *and* $\vec{b}$.

SOLUTION We have

$$(\vec{a} \times \vec{b}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 3 \\ 2 & 3 & -5 \end{vmatrix}$$
$$= (10 - 9)\,\hat{i} - (-5 - 6)\,\hat{j} + (3 + 4)\,\hat{k} = (\hat{i} + 11\hat{j} + 7\hat{k}).$$

Now, $(\vec{a} \times \vec{b}) \cdot \vec{a} = (\hat{i} + 11\,\hat{j} + 7\,\hat{k}) \cdot (\hat{i} - 2\,\hat{j} + 3\,\hat{k})$.

$\therefore \quad (\vec{a} \times \vec{b}) \cdot \vec{a} = (1 - 22 + 21) = 0.$

$\therefore \quad (\vec{a} \times \vec{b}) \perp \vec{a}.$

And, $(\vec{a} \times \vec{b}) \cdot \vec{b} = (\hat{i} + 11\,\vec{j} + 7\,\vec{k}) \cdot (2\hat{i} + 3\,\hat{j} - 5\,\hat{k})$

$= (2 + 33 - 35) = 0.$

$\therefore \quad (\vec{a} \times \vec{b}) \perp \vec{b}.$

EXAMPLE 3 *If* $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ *and* $\vec{b} = \hat{j} - \hat{k}$, *find a vector* $\vec{c}$ *such that* $\vec{a} \times \vec{c} = \vec{b}$ *and* $\vec{a} \cdot \vec{c} = 3$. **[CBSE 2013]**

SOLUTION Here $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{b} = \hat{j} - \hat{k}$.

Let $\vec{c} = c_1\,\hat{i} + c_2\,\hat{j} + c_3\,\hat{k}$. Then,

$\vec{a} \cdot \vec{c} = 3$ and $\vec{a} \times \vec{c} = \vec{b}$

$\Rightarrow \quad (\hat{i} + \hat{j} + \hat{k}) \cdot (c_1\,\hat{i} + c_2\,\hat{j} + c_3\,\hat{k}) = 3$

and $(\hat{i} + \hat{j} + \hat{k}) \times (c_1\,\hat{i} + c_2\,\hat{j} + c_3\,\hat{k}) = (\hat{j} - \hat{k})$

$$\Rightarrow \quad c_1 + c_2 + c_3 = 3 \quad \text{... (i)} \quad \text{and} \quad \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ c_1 & c_2 & c_3 \end{vmatrix} = (\hat{j} - \hat{k}) \quad \text{... (ii)}$$

Now, (ii) gives: $(c_3 - c_2)\,\hat{i} - (c_3 - c_1)\,\hat{j} + (c_2 - c_1)\,\hat{k} = (\hat{j} - \hat{k})$

$\Rightarrow \quad c_3 - c_2 = 0,\ c_1 - c_3 = 1$ and $c_2 - c_1 = -1$

$\Rightarrow \quad c_3 = c_2$ and $c_1 - c_2 = 1.$

Putting $c_3 = c_2$ in (i), we get $c_1 + 2c_2 = 3$.

On solving $c_1 + 2c_2 = 3$ and $c_1 - c_2 = 1$, we get $c_2 = \frac{2}{3}$ and $c_1 = \frac{5}{3}$.

$\therefore \quad c_1 = \frac{5}{3},\ c_2 = \frac{2}{3}$ and $c_3 = \frac{2}{3}$.

Hence, $\vec{c} = \frac{5}{3}\,\hat{i} + \frac{2}{3}\,\hat{j} + \frac{2}{3}\,\hat{k} \ \Rightarrow \ \vec{c} = \frac{1}{3}(5\,\hat{i} + 2\,\hat{j} + 2\,\hat{k})$.

EXAMPLE 4 *Find a unit vector perpendicular to each one of the vectors* $\vec{a} = (4\,\hat{i} - \hat{j} + 3\,\hat{k})$ *and* $\vec{b} = (2\,\hat{i} + 2\,\hat{j} - \hat{k})$.

SOLUTION We know that $(\vec{a} \times \vec{b})$ is a vector perpendicular to each one of $\vec{a}$ and $\vec{b}$. So, the required vector is $\frac{(\vec{a} \times \vec{b})}{|\vec{a} \times \vec{b}|}$.

$$\text{Now, } (\vec{a} \times \vec{b}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & -1 & 3 \\ 2 & 2 & -1 \end{vmatrix}$$

$$= (1-6)\hat{i} - (-4-6)\hat{j} + (8+2)\hat{k}$$

$$= (-5\hat{i} + 10\hat{j} + 10\hat{k}).$$

$$|\vec{a} \times \vec{b}| = \sqrt{(-5)^2 + (10)^2 + (10)^2} = \sqrt{225} = 15.$$

$$\text{Hence, the required unit vector} = \frac{(-5\hat{i} + 10\hat{j} + 10\hat{k})}{15}$$

$$= \frac{1}{3}(-\hat{i} + 2\hat{j} + 2\hat{k}).$$

EXAMPLE 5 *Find a vector of magnitude* 15, *which is perpendicular to both the vectors* $(4\hat{i} - \hat{j} + 8\hat{k})$ *and* $(-\hat{j} + \hat{k})$.

SOLUTION Let $\vec{a} = (4\hat{i} - \hat{j} + 8\hat{k})$ and $\vec{b} = (-\hat{j} + \hat{k})$.

A unit vector perpendicular to both $\vec{a}$ and $\vec{b} = \frac{(\vec{a} \times \vec{b})}{|\vec{a} \times \vec{b}|}$.

$$\text{Now, } \vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & -1 & 8 \\ 0 & -1 & 1 \end{vmatrix}$$

$$= (-1+8)\hat{i} - (4-0)\hat{j} + (-4-0)\hat{k}$$

$$= (7\hat{i} - 4\hat{j} - 4\hat{k}).$$

$$\therefore \quad |\vec{a} \times \vec{b}| = \sqrt{7^2 + (-4)^2 + (-4)^2} = \sqrt{81} = 9.$$

So, a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$

$$= \frac{(\vec{a} \times \vec{b})}{|\vec{a} \times \vec{b}|} = \frac{7\hat{i} - 4\hat{j} - 4\hat{k}}{9}.$$

$$\text{The required vector} = \frac{15(7\hat{i} - 4\hat{j} - 4\hat{k})}{9} = \frac{5}{3}(7\hat{i} - 4\hat{j} - 4\hat{k}).$$

EXAMPLE 6 *Let* $\vec{a}=(\hat{i}+4\hat{j}+2\hat{k}),\ \vec{b}=(3\hat{i}-2\hat{j}+7\hat{k})$ *and* $\vec{c}=(2\hat{i}-\hat{j}+4\hat{k})$. *Find a vector* $\vec{d}$ *which is perpendicular to both* $\vec{a}$ *and* $\vec{b}$ *such that* $\vec{c}\cdot\vec{d}=18$. **[CBSE 2010]**

SOLUTION Since $\vec{d}$ is perpendicular to both $\vec{a}$ and $\vec{b}$, it follows that $\vec{d}$ is parallel to $(\vec{a}\times\vec{b})$.

$$\text{Now, } (\vec{a}\times\vec{b})=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 4 & 2\\ 3 & -2 & 7\end{vmatrix}$$

$$=(28+4)\hat{i}-(7-6)\hat{j}+(-2-12)\hat{k}=(32\hat{i}-\hat{j}-14\hat{k}).$$

Since $\vec{d}$ is parallel to $(\vec{a}\times\vec{b})$, we have $\vec{d}=\lambda(\vec{a}\times\vec{b})$ for some scalar λ.

$$\therefore\quad \vec{d}=\lambda(32\hat{i}-\hat{j}-14\hat{k})=(32\lambda\hat{i}-\lambda\hat{j}-14\lambda\hat{k}) \qquad \text{... (i)}$$

Since $\vec{c}\cdot\vec{d}=18$, we have

$$(2\hat{i}-\hat{j}+4\hat{k})\cdot(32\lambda\hat{i}-\lambda\hat{j}-14\lambda\hat{k})=18$$

$$\Rightarrow\quad (64\lambda+\lambda-56\lambda)=18\Rightarrow 9\lambda=18\Rightarrow \lambda=2.$$

Hence, $\vec{d}=(64\hat{i}-2\hat{j}-28\hat{k})$ [putting $\lambda=2$ in (i)].

EXAMPLE 7 *Find a vector of magnitude* 5 *units, perpendicular to each of the vectors* $(\vec{a}+\vec{b})$ *and* $(\vec{a}-\vec{b})$, *where* $\vec{a}=(\hat{i}+\hat{j}+\hat{k})$ *and* $\vec{b}=(\hat{i}+2\hat{j}+3\hat{k})$. **[CBSE 2008C]**

SOLUTION We have

$$(\vec{a}+\vec{b})=(\hat{i}+\hat{j}+\hat{k})+(\hat{i}+2\hat{j}+3\hat{k})=(2\hat{i}+3\hat{j}+4\hat{k}) \text{ and}$$

$$(\vec{a}-\vec{b})=(\hat{i}+\hat{j}+\hat{k})-(\hat{i}+2\hat{j}+3\hat{k})=(-\hat{j}-2\hat{k}).$$

$$\therefore\quad (\vec{a}+\vec{b})\times(\vec{a}-\vec{b})=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 3 & 4\\ 0 & -1 & -2\end{vmatrix}$$

$$=(-6+4)\hat{i}-(-4-0)\hat{j}+(-2-0)\hat{k}$$

$$=(-2\hat{i}+4\hat{j}-2\hat{k}).$$

$$\therefore\quad |(\vec{a}+\vec{b})\times(\vec{a}-\vec{b})|=\sqrt{(-2)^2+4^2+(-2)^2}=\sqrt{24}=2\sqrt{6}.$$

So, the vectors of magnitude 5 units and perpendicular to each of the vectors $(\vec{a}+\vec{b})$ and $(\vec{a}-\vec{b})$ are:

$$\pm\frac{5\{(\vec{a}+\vec{b})\times(\vec{a}-\vec{b})\}}{|(\vec{a}+\vec{b})\times(\vec{a}-\vec{b})|}=\pm\frac{5(-2\hat{i}+4\hat{j}-2\hat{k})}{2\sqrt{6}}=\pm\frac{5(-\hat{i}+2\hat{j}-\hat{k})}{\sqrt{6}}.$$

EXAMPLE 8 *If* $\vec{a}=4\hat{i}+3\hat{j}+2\hat{k}$ *and* $\vec{b}=3\hat{i}+2\hat{k}$, *find* $|\vec{b}\times 2\vec{a}|$.

SOLUTION We have

$$\vec{b}=(3\hat{i}+2\hat{k}) \text{ and } 2\vec{a}=(8\hat{i}+6\hat{j}+4\hat{k}).$$

$$\therefore\quad (\vec{b}\times 2\vec{a})=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\3&0&2\\8&6&4\end{vmatrix}$$

$$=(0-12)\hat{i}+(16-12)\hat{j}+(18-0)\hat{k}$$

$$=(-12\hat{i}+4\hat{j}+18\hat{k}).$$

$$\therefore\quad |\vec{b}\times 2\vec{a}|=|-12\hat{i}+4\hat{j}+18\hat{k}|$$

$$=\sqrt{(-12)^2+4^2+(18)^2}=\sqrt{484}=22.$$

Hence, $|\vec{b}\times 2\vec{a}|=22.$

EXAMPLE 9 *If* $|\vec{a}|=\sqrt{26}, |\vec{b}|=7$ *and* $|\vec{a}\times\vec{b}|=35$, *find* $\vec{a}\cdot\vec{b}$.

SOLUTION Given that $|\vec{a}|=\sqrt{26}$ and $|\vec{b}|=7$, and $|\vec{a}\times\vec{b}|=35.$

$$\therefore\quad |\vec{a}\times\vec{b}|=35 \Rightarrow |\vec{a}||\vec{b}|\sin\theta=35$$

$$\Rightarrow \sin\theta=\frac{35}{|\vec{a}||\vec{b}|}=\frac{35}{(\sqrt{26})\times 7}=\frac{5}{\sqrt{26}}.$$

Now, $\cos\theta=\sqrt{1-\sin^2\theta}=\sqrt{1-\frac{25}{26}}=\frac{1}{\sqrt{26}}.$

$$\therefore\quad \vec{a}\cdot\vec{b}=|\vec{a}||\vec{b}|\cos\theta=\left(\sqrt{26}\times 7\times\frac{1}{\sqrt{26}}\right)=7.$$

Hence, $\vec{a}\cdot\vec{b}=7.$

EXAMPLE 10 *If* $|\vec{a}|=2, |\vec{b}|=7$ *and* $(\vec{a}\times\vec{b})=(3\hat{i}+2\hat{j}+6\hat{k})$, *find the angle between* $\vec{a}$ *and* $\vec{b}$.

SOLUTION Let 0 be the angle between $\vec{a}$ and $\vec{b}$. Then,

$$\vec{a} \times \vec{b} = 3\hat{i} + 2\hat{j} + 6\hat{k}$$

$$\Rightarrow \quad |\vec{a} \times \vec{b}| = \sqrt{3^2 + 2^2 + 6^2} = \sqrt{49} = 7$$

$$\Rightarrow \quad |\vec{a}||\vec{b}| \sin\theta = 7 \qquad [\because \quad |\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\theta]$$

$$\Rightarrow \quad \sin\theta = \frac{7}{|\vec{a}||\vec{b}|} = \frac{7}{(2 \times 7)} = \frac{1}{2} \quad \Rightarrow \quad \theta = \frac{\pi}{6}.$$

Hence, the required angle between $\vec{a}$ and $\vec{b}$ is $\frac{\pi}{6}$.

EXAMPLE 11 *Find the sine of the angle between the vectors* $\vec{a} = (2\hat{i} - \hat{j} + 3\hat{k})$ *and* $\vec{b} = (\hat{i} + 3\hat{j} + 2\hat{k})$.

SOLUTION We have

$$(\vec{a} \times \vec{b}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 3 \\ 1 & 3 & 2 \end{vmatrix}$$

$$= (-2 - 9)\hat{i} - (4 - 3)\hat{j} + (6 + 1)\hat{k}$$

$$= (-11\hat{i} - \hat{j} + 7\hat{k}).$$

$$|\vec{a} \times \vec{b}| = \sqrt{(-11)^2 + (-1)^2 + 7^2} = \sqrt{171} = 3\sqrt{19},$$

$$|\vec{a}| = \sqrt{2^2 + (-1)^2 + 3^2} = \sqrt{14},$$

$$|\vec{b}| = \sqrt{1^2 + 3^2 + 2^2} = \sqrt{14}.$$

Let θ be the angle between $\vec{a}$ and $\vec{b}$. Then,

$$\sin\theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}||\vec{b}|} = \frac{3\sqrt{19}}{(\sqrt{14})(\sqrt{14})} = \frac{3}{14}\sqrt{19}.$$

EXAMPLE 12 *If the vectors* $\vec{a}$ *and* $\vec{b}$ *are such that* $|\vec{a}| = 3$, $|\vec{b}| = \frac{2}{3}$ *and* $\vec{a} \times \vec{b}$ *is a unit vector then write the angle between* $\vec{a}$ *and* $\vec{b}$. **[CBSE 2014]**

SOLUTION Let θ be the angle between $\vec{a}$ and $\vec{b}$. Then,

$$|\vec{a} \times \vec{b}| = 1 \Rightarrow |\vec{a}||\vec{b}| \sin\theta = 1 \Rightarrow \left(3 \times \frac{2}{3}\right) \sin\theta = 1$$

$$\Rightarrow \quad \sin\theta = \frac{1}{2} \Rightarrow \theta = 30°.$$

Hence, the angle between $\vec{a}$ and $\vec{b}$ is $30°$.

EXAMPLE 13 *Find the area of the parallelogram whose adjacent sides are represented by the vectors* $(3\hat{i}+\hat{j}-2\hat{k})$ *and* $(\hat{i}-3\hat{j}+4\hat{k})$.

SOLUTION Let $\vec{a}=(3\hat{i}+\hat{j}-2\hat{k})$ and $\vec{b}=(\hat{i}-3\hat{j}+4\hat{k})$.

Then, vector area of the ||gm is $(\vec{a}\times\vec{b})$.

$$\text{Now, } (\vec{a}\times\vec{b})=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 3 & 1 & -2\\ 1 & -3 & 4\end{vmatrix}$$

$$=(4-6)\hat{i}-(12+2)\hat{j}+(-9-1)\hat{k}$$

$$=(-2\hat{i}-14\hat{j}-10\hat{k}).$$

$$\text{Required area}=|\vec{a}\times\vec{b}|$$

$$=\sqrt{(-2)^2+(-14)^2+(-10)^2}\text{ sq units}$$

$$=\sqrt{300}\text{ sq units}=10\sqrt{3}\text{ sq units.}$$

EXAMPLE 14 *Find the area of the parallelogram whose diagonals are represented by the vectors* $\vec{d}_1=(2\hat{i}-\hat{j}+\hat{k})$ *and* $\vec{d}_2=(3\hat{i}+4\hat{j}-\hat{k})$.

SOLUTION Given that $\vec{d}_1=(2\hat{i}-\hat{j}+\hat{k})$ and $\vec{d}_2=(3\hat{i}+4\hat{j}-\hat{k})$.

Vector area of the ||gm is $\frac{1}{2}(\vec{d}_1\times\vec{d}_2)$.

$$\text{Now, } (\vec{d}_1\times\vec{d}_2)=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & -1 & 1\\ 3 & 4 & -1\end{vmatrix}$$

$$=(1-4)\hat{i}-(-2-3)\hat{j}+(8+3)\hat{k}$$

$$=(-3\hat{i}+5\hat{j}+11\hat{k}).$$

$$\text{Required area}=\frac{1}{2}|\vec{d}_1\times\vec{d}_2|$$

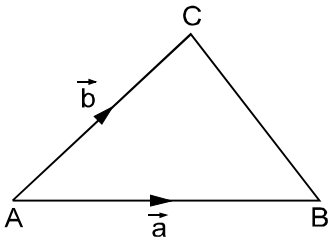
$$=\frac{1}{2}\sqrt{(-3)^2+5^2+(11)^2}\text{ sq units}$$

$$=\frac{1}{2}\sqrt{155}\text{ sq units.}$$

EXAMPLE 15 *Find the area of the triangle whose adjacent sides are determined by the vectors* $\vec{a}=(-2\hat{i}-5\hat{k})$ *and* $\vec{b}=(\hat{i}-2\hat{j}-\hat{k})$.

SOLUTION Two adjacent sides of the given triangle are represented by the vectors $\vec{a} = (-2\hat{i} - 5\hat{k})$ and $\vec{b} = (\hat{i} - 2\hat{j} - \hat{k})$.

So, the area of the triangle is $\frac{1}{2}|\vec{a} \times \vec{b}|$.



$$\text{Now, } (\vec{a} \times \vec{b}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 0 & -5 \\ 1 & -2 & -1 \end{vmatrix}$$

$$= (0 - 10)\hat{i} - (2 + 5)\hat{j} + (4 - 0)\hat{k}$$

$$= (-10\hat{i} - 7\hat{j} + 4\hat{k}).$$

$$\therefore \text{ required area} = \frac{1}{2}|\vec{a} \times \vec{b}|$$

$$= \frac{1}{2} \cdot \{\sqrt{(-10)^2 + (-7)^2 + 4^2}\} \text{ sq units}$$

$$= \frac{1}{2}\sqrt{165} \text{ sq units.}$$

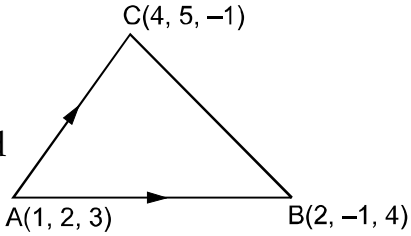
EXAMPLE 16 *Using vectors find the area of* $\triangle ABC$ *whose vertices are* $A(1, 2, 3)$, $B(2, -1, 4)$ *and* $C(4, 5, -1)$. **[CBSE 2013]**

SOLUTION We have

Position vector of $A = (\hat{i} + 2\hat{j} + 3\hat{k})$,

position vector of $B = (2\hat{i} - \hat{j} + 4\hat{k})$ and

position vector of $C = (4\hat{i} + 5\hat{j} - \hat{k})$.



$$\therefore \quad \overrightarrow{AB} = (\text{position vector of } B) - (\text{position vector of } A)$$

$$= (2\hat{i} - \hat{j} + 4\hat{k}) - (\hat{i} + 2\hat{j} + 3\hat{k}) = (\hat{i} - 3\hat{j} + \hat{k}).$$

$$\overrightarrow{AC} = (\text{position vector of } C) - (\text{position vector of } A)$$

$$= (4\hat{i} + 5\hat{j} - \hat{k}) - (\hat{i} + 2\hat{j} + 3\hat{k}) = (3\hat{i} + 3\hat{j} - 4\hat{k}).$$

$$\therefore \quad \text{area of } \triangle ABC = \left|\frac{1}{2}(\overrightarrow{AB} \times \overrightarrow{AC})\right|.$$

$$\text{Now, } \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -3 & 1 \\ 3 & 3 & -4 \end{vmatrix}$$

$$= (12-3)\,\hat{i} - (-4-3)\,\hat{j} + (3+9)\,\hat{k}$$

$$= (9\hat{i} + 7\hat{j} + 12\hat{k}).$$

$$\therefore \quad \text{area of } \Delta ABC = \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}|$$

$$= \frac{1}{2}\,|(9\hat{i} + 7\hat{j} + 12\hat{k})| = \frac{1}{2}\cdot\sqrt{(9)^2 + 7^2 + (12)^2}$$

$$= \frac{1}{2}\cdot\sqrt{(81+49+144)} = \frac{1}{2}\sqrt{274} \text{ sq units.}$$

Hence, the area of ΔABC is $\frac{1}{2}\sqrt{274}$ sq units.

EXAMPLE 17 *Show that the points whose position vectors are* $(5\hat{i} + 6\hat{j} + 7\hat{k})$, $(7\hat{i} - 8\hat{j} + 9\hat{k})$ *and* $(3\hat{i} + 20\hat{j} + 5\hat{k})$ *are collinear.*

SOLUTION Let the given points be A, B, and C respectively. Then,

$$\overrightarrow{AB} = (\text{position vector of } B) - (\text{position vector of } A)$$

$$= (7\hat{i} - 8\hat{j} + 9\hat{k}) - (5\hat{i} + 6\hat{j} + 7\hat{k}) = (2\hat{i} - 14\hat{j} + 2\hat{k}).$$

And, $\overrightarrow{AC} = (\text{position vector of } C) - (\text{position vector of } A)$

$$= (3\hat{i} + 20\hat{j} + 5\hat{k}) - (5\hat{i} + 6\hat{j} + 7\hat{k})$$

$$= (-2\hat{i} + 14\hat{j} - 2\hat{k}).$$

$$\therefore \quad (\overrightarrow{AB} \times \overrightarrow{AC}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -14 & 2 \\ -2 & 14 & -2 \end{vmatrix}$$

$$= 2\times(-2)\cdot\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -7 & 1 \\ 1 & -7 & 1 \end{vmatrix} = (-4)\times\vec{0} = \vec{0}$$

[$\because$ R_2 and R_3 are identical].

So, $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are parallel vectors, having a common end point, A. Hence, the given points A, B and C are collinear.

EXAMPLE 18 *Using vector method, show that the points* $A(2, -1, 3)$, $B(4, 3, 1)$ *and* $C(3, 1, 2)$ *are collinear.*

SOLUTION Clearly, we have

position vector of $A = (2\hat{i} - \hat{j} + 3\hat{k})$,

position vector of $B = (4\hat{i} + 3\hat{j} + \hat{k})$, and

position vector of $C = (3\hat{i} + \hat{j} + 2\hat{k})$.

$\therefore \quad \overrightarrow{AB} = (\text{position vector of } B) - (\text{position vector of } A)$

$= (4\hat{i} + 3\hat{j} + \hat{k}) - (2\hat{i} - \hat{j} + 3\hat{k}) = (2\hat{i} + 4\hat{j} - 2\hat{k}).$

$\overrightarrow{AC} = (\text{position vector of } C) - (\text{position vector of } A)$

$= (3\hat{i} + \hat{j} + 2\hat{k}) - (2\hat{i} - \hat{j} + 3\hat{k}) = (\hat{i} + 2\hat{j} - \hat{k}).$

$\therefore \quad (\overrightarrow{AB} \times \overrightarrow{AC}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 4 & -2 \\ 1 & 2 & -1 \end{vmatrix}$

$= 2\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -1 \\ 1 & 2 & -1 \end{vmatrix} = \vec{0}$ [$\because$ R_2 and R_3 are identical].

Thus, $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are parallel vectors, having a common end point, A.

Hence, the given points A, B, and C are collinear.

EXAMPLE 19 *Show that the points having position vectors*

$$(\vec{a} - 2\vec{b} + 3\vec{c}), (-2\vec{a} + 3\vec{b} + 2\vec{c}), (-8\vec{a} + 13\vec{b})$$

are collinear, whatever be $\vec{a}, \vec{b}, \vec{c}$.

SOLUTION Let A, B, C be the given points whose position vectors are $(\vec{a} - 2\vec{b} + 3\vec{c})$, $(-2\vec{a} + 3\vec{b} + 2\vec{c})$ and $(-8\vec{a} + 13\vec{b})$ respectively. Then,

$\overrightarrow{AB} = (\text{position vector of } B) - (\text{position vector of } A)$

$= (-2\vec{a} + 3\vec{b} + 2\vec{c}) - (\vec{a} - 2\vec{b} + 3\vec{c}) = (-3\vec{a} + 5\vec{b} - \vec{c})$, and

$\overrightarrow{AC} = (\text{position vector of } C) - (\text{position vector of } A)$

$= (-8\vec{a} + 13\vec{b}) - (\vec{a} - 2\vec{b} + 3\vec{c}) = (-9\vec{a} + 15\vec{b} - 3\vec{c}).$

$\therefore \quad (\overrightarrow{AB} \times \overrightarrow{AC}) = (-3\vec{a} + 5\vec{b} - \vec{c}) \times (-9\vec{a} + 15\vec{b} - 3\vec{c})$

$= \vec{p} \times 3\vec{p}$, where $(-3\vec{a} + 5\vec{b} - \vec{c}) = \vec{p}$

$= \vec{0}$ [$\because$ $\vec{p} \times \vec{p} = \vec{0}$].

$\therefore \overrightarrow{AB}$ and $\overrightarrow{AC}$ are parallel vectors, having a common end point, A.

Hence, the points A, B and C are collinear.

EXAMPLE 20 *Prove that* $(\vec{a}-\vec{b})\times(\vec{a}+\vec{b})=2(\vec{a}\times\vec{b})$.

PROOF We have

$$(\vec{a}-\vec{b})\times(\vec{a}+\vec{b})$$
$$=\vec{a}\times\vec{a}+\vec{a}\times\vec{b}-\vec{b}\times\vec{a}-\vec{b}\times\vec{b} \quad \text{[by the distributive law]}$$
$$=\vec{a}\times\vec{b}-\vec{b}\times\vec{a} \qquad [\because\ \vec{a}\times\vec{a}=\vec{0} \text{ and } \vec{b}\times\vec{b}=\vec{0}\,]$$
$$=(\vec{a}\times\vec{b})+(\vec{a}\times\vec{b}) \qquad [\because\ -\vec{b}\times\vec{a}=(\vec{a}\times\vec{b})]$$
$$=2(\vec{a}\times\vec{b}).$$

Hence, $(\vec{a}-\vec{b})\times(\vec{a}+\vec{b})=2(\vec{a}\times\vec{b})$.

EXAMPLE 21 *If* $\vec{a}\times\vec{b}=\vec{c}\times\vec{d}$ *and* $\vec{a}\times\vec{c}=\vec{b}\times\vec{d}$, *show that* $(\vec{a}-\vec{d})$ *is parallel to* $(\vec{b}-\vec{c})$, *it being given that* $a\neq d$ *and* $b\neq c$. **[CBSE 2009]**

SOLUTION $\vec{a}\times\vec{b}=\vec{c}\times\vec{d}$, and $\vec{a}\times\vec{c}=\vec{b}\times\vec{d}$

$$\Rightarrow \vec{a}\times\vec{b}-\vec{a}\times\vec{c}=\vec{c}\times\vec{d}-\vec{b}\times\vec{d}$$
$$\Rightarrow \vec{a}\times\vec{b}-\vec{a}\times\vec{c}+\vec{b}\times\vec{d}-\vec{c}\times\vec{d}=\vec{0}$$
$$\Rightarrow \vec{a}\times(\vec{b}-\vec{c})+(\vec{b}-\vec{c})\times\vec{d}=\vec{0}$$
$$\Rightarrow \vec{a}\times(\vec{b}-\vec{c})-\vec{d}\times(\vec{b}-\vec{c})=\vec{0}$$
$$\Rightarrow (\vec{a}-\vec{d})\times(\vec{b}-\vec{c})=\vec{0}$$
$$\Rightarrow (\vec{a}-\vec{d})\,||\,(\vec{b}-\vec{c}).$$

Hence, $(\vec{a}-\vec{d})$ is parallel to $(\vec{b}-\vec{c})$.

EXAMPLE 22 *Prove that* $\vec{a}\times(\vec{b}+\vec{c})+\vec{b}\times(\vec{c}+\vec{a})+\vec{c}\times(\vec{a}+\vec{b})=\vec{0}$.

SOLUTION We have

$$\vec{a}\times(\vec{b}+\vec{c})+\vec{b}\times(\vec{c}+\vec{a})+\vec{c}\times(\vec{a}+\vec{b})$$
$$=(\vec{a}\times\vec{b})+(\vec{a}\times\vec{c})+(\vec{b}\times\vec{c})+(\vec{b}\times\vec{a})+(\vec{c}\times\vec{a})+(\vec{c}\times\vec{b})$$
[by the distributive law]
$$=(\vec{a}\times\vec{b})+(\vec{b}\times\vec{c})+(\vec{c}\times\vec{a})-(\vec{a}\times\vec{b})-(\vec{b}\times\vec{c})-(\vec{c}\times\vec{a})$$
$$=\vec{0} \quad [\because\ (\vec{b}\times\vec{a})=-(\vec{a}\times\vec{b}),\ (\vec{c}\times\vec{b})=-(\vec{b}\times\vec{c})$$
and $(\vec{a}\times\vec{c})=-(\vec{c}\times\vec{a})$].

Hence, $\vec{a}\times(\vec{b}+\vec{c})+\vec{b}\times(\vec{c}+\vec{a})+\vec{c}\times(\vec{a}+\vec{b})=\vec{0}$.

EXAMPLE 23 *If* $\vec{a}+\vec{b}+\vec{c}=\vec{0}$, *prove that* $(\vec{a}\times\vec{b})=(\vec{b}\times\vec{c})=(\vec{c}\times\vec{a})$. **[CBSE 2001, '04C]**

SOLUTION $\vec{a}+\vec{b}+\vec{c}=\vec{0} \Rightarrow \vec{a}+\vec{b}=-\vec{c}$

$\Rightarrow (\vec{a}+\vec{b})\times\vec{b}=(-\vec{c})\times\vec{b}$

$\Rightarrow (\vec{a}\times\vec{b})+(\vec{b}\times\vec{b})=(-\vec{c})\times\vec{b}$

[by the distributive law]

$\Rightarrow (\vec{a}\times\vec{b})+\vec{0}=(\vec{b}\times\vec{c})$

[$\because \vec{b}\times\vec{b}=\vec{0}$ and $(-\vec{c})\times\vec{b}=\vec{b}\times\vec{c}$]

$\Rightarrow \vec{a}\times\vec{b}=\vec{b}\times\vec{c}$ … (i)

Also, $\vec{a}+\vec{b}+\vec{c}=\vec{0} \Rightarrow \vec{b}+\vec{c}=-\vec{a}$

$\Rightarrow (\vec{b}+\vec{c})\times\vec{c}=(-\vec{a})\times\vec{c}$

$\Rightarrow (\vec{b}\times\vec{c})+(\vec{c}\times\vec{c})=(-\vec{a})\times\vec{c}$

[by the distributive law]

$\Rightarrow (\vec{b}\times\vec{c})+\vec{0}=\vec{c}\times\vec{a}$

[$\because \vec{c}\times\vec{c}=\vec{0}$ and $(-\vec{a})\times\vec{c}=\vec{c}\times\vec{a}$]

$\Rightarrow \vec{b}\times\vec{c}=\vec{c}\times\vec{a}$ … (ii)

From (i) and (ii), we get $\vec{a}\times\vec{b}=\vec{b}\times\vec{c}=\vec{c}\times\vec{a}$.

EXAMPLE 24 *Prove that the points A, B, C with position vectors* $\vec{a}, \vec{b}, \vec{c}$ *are collinear if and only if* $(\vec{b}\times\vec{c})+(\vec{c}\times\vec{a})+(\vec{a}\times\vec{b})=\vec{0}$.

PROOF We have

$\overrightarrow{AB}=$ (position vector of B) $-$ (position vector of A) $=(\vec{b}-\vec{a})$

and $\overrightarrow{BC}=$ (position vector of C) $-$ (position vector of B) $=(\vec{c}-\vec{b})$.

Now, A, B, C are collinear

$\Leftrightarrow$ $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are parallel $\Leftrightarrow$ $(\vec{b}-\vec{a})\times(\vec{c}-\vec{b})=\vec{0}$

$\Leftrightarrow (\vec{b}-\vec{a})\times\vec{c}-(\vec{b}-\vec{a})\times\vec{b}=\vec{0}$ [by the distributive law]

$\Leftrightarrow \vec{b}\times\vec{c}-\vec{a}\times\vec{c}-\vec{b}\times\vec{b}+\vec{a}\times\vec{b}=\vec{0}$ [by the distributive law]

$\Leftrightarrow (\vec{b}\times\vec{c})+(\vec{c}\times\vec{a})+(\vec{a}\times\vec{b})=\vec{0}$ [$\because \vec{b}\times\vec{b}=\vec{0}$ and $-\vec{a}\times\vec{c}=\vec{c}\times\vec{a}$].

Thus, A, B, C are collinear $\Leftrightarrow (\vec{b}\times\vec{c})+(\vec{c}\times\vec{a})+(\vec{a}\times\vec{b})=\vec{0}$.

EXAMPLE 25 *If* $\vec{a}\cdot\vec{b}=0$ *and* $\vec{a}\times\vec{b}=\vec{0}$, *prove that* $\vec{a}=\vec{0}$ *or* $\vec{b}=\vec{0}$.

PROOF Let $\vec{a}\cdot\vec{b}=0$ and $\vec{a}\times\vec{b}=\vec{0}$. Then,

$$\vec{a}\cdot\vec{b}=0 \text{ and } \vec{a}\times\vec{b}=\vec{0}$$

$$\Rightarrow \quad (\vec{a}=\vec{0} \text{ or } \vec{b}=\vec{0} \text{ or } \vec{a}\perp\vec{b}) \text{ and } (\vec{a}=\vec{0} \text{ or } \vec{b}=\vec{0} \text{ or } \vec{a}\,||\,\vec{b})$$

$$\Rightarrow \quad \vec{a}=\vec{0} \text{ or } \vec{b}=\vec{0}$$

[$\because$ $\vec{a}\perp\vec{b}$ and $\vec{a}\,||\,\vec{b}$ can never hold simultaneously].

Hence, $(\vec{a}\cdot\vec{b}=0 \text{ and } \vec{a}\times\vec{b}=\vec{0}) \Rightarrow (\vec{a}=\vec{0} \text{ or } \vec{b}=\vec{0})$.

EXAMPLE 26 *If* $\vec{a}\cdot\vec{b}=\vec{a}\cdot\vec{c}$, $\vec{a}\times\vec{b}=\vec{a}\times\vec{c}$ *and* $\vec{a}\neq\vec{0}$ *then prove that* $\vec{b}=\vec{c}$.

PROOF $\vec{a}\cdot\vec{b}=\vec{a}\cdot\vec{c}$ and $\vec{a}\neq\vec{0}$

$$\Rightarrow \quad \vec{a}\cdot\vec{b}-\vec{a}\cdot\vec{c}=0 \text{ and } \vec{a}\neq\vec{0} \quad \Rightarrow \quad \vec{a}\cdot(\vec{b}-\vec{c})=0 \text{ and } \vec{a}\neq\vec{0}$$

$$\Rightarrow \quad \vec{b}-\vec{c}=\vec{0} \text{ or } \vec{a}\perp(\vec{b}-\vec{c}) \quad \Rightarrow \quad \vec{b}=\vec{c} \text{ or } \vec{a}\perp(\vec{b}-\vec{c}). \qquad \text{... (i)}$$

Again, $\vec{a}\times\vec{b}=\vec{a}\times\vec{c}$ and $\vec{a}\neq\vec{0}$

$$\Rightarrow \quad (\vec{a}\times\vec{b})-(\vec{a}\times\vec{c})=\vec{0} \text{ and } \vec{a}\neq\vec{0} \Rightarrow \quad \vec{a}\times(\vec{b}-\vec{c})=\vec{0} \text{ and } \vec{a}\neq\vec{0}$$

$$\Rightarrow \quad (\vec{b}-\vec{c})=\vec{0} \text{ or } \vec{a}\,||\,(\vec{b}-\vec{c}) \quad \Rightarrow \quad \vec{b}=\vec{c} \text{ or } \vec{a}\,||\,(\vec{b}-\vec{c}). \qquad \text{... (ii)}$$

From (i) and (ii), we get $\vec{b}=\vec{c}$

[$\because$ $\vec{a}\perp(\vec{b}-\vec{c})$ and $\vec{a}\,||\,(\vec{b}-\vec{c})$ both cannot hold simultaneously].

Lagrange's Identity

EXAMPLE 27 *Prove that* $|\vec{a}\times\vec{b}|^2=\begin{vmatrix}\vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b}\\ \vec{a}\cdot\vec{b} & \vec{b}\cdot\vec{b}\end{vmatrix}$. **[CBSE 2002C, '04]**

SOLUTION Let θ be the angle betwen $\vec{a}$ and $\vec{b}$. Then,

$$|\vec{a}\times\vec{b}|^2=(\vec{a}\times\vec{b})\cdot(\vec{a}\times\vec{b})=(ab\sin\theta)\hat{n}\cdot(ab\sin\theta)\hat{n}$$

$$=(a^2b^2\sin^2\theta)(\hat{n}\cdot\hat{n})=a^2b^2\sin^2\theta$$

$$=a^2b^2(1-\cos^2\theta)=a^2b^2-(ab\cos\theta)^2$$

$$=(\vec{a}\cdot\vec{a})(\vec{b}\cdot\vec{b})-(\vec{a}\cdot\vec{b})^2=\begin{vmatrix}\vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b}\\ \vec{a}\cdot\vec{b} & \vec{b}\cdot\vec{b}\end{vmatrix}.$$

Hence, $|\vec{a}\times\vec{b}|^2=\begin{vmatrix}\vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b}\\ \vec{a}\cdot\vec{b} & \vec{b}\cdot\vec{b}\end{vmatrix}$.

EXERCISE 24

1. Find $(\vec{a} \times \vec{b})$ and $|\vec{a} \times \vec{b}|$, when
   (i) $\vec{a} = \hat{i} - \hat{j} + 2\hat{k}$ and $\vec{b} = 2\hat{i} + 3\hat{j} - 4\hat{k}$
   (ii) $\vec{a} = 2\hat{i} + \hat{j} + 3\hat{k}$ and $\vec{b} = 3\hat{i} + 5\hat{j} - 2\hat{k}$
   (iii) $\vec{a} = \hat{i} - 7\hat{j} + 7\hat{k}$ and $\vec{b} = 3\hat{i} - 2\hat{j} + 2\hat{k}$
   (iv) $\vec{a} = 4\hat{i} + \hat{j} - 2\hat{k}$ and $\vec{b} = 3\hat{i} + \hat{k}$
   (v) $\vec{a} = 3\hat{i} + 4\hat{j}$ and $\vec{b} = \hat{i} + \hat{j} + \hat{k}$

2. Find λ if $(2\hat{i} + 6\hat{j} + 14\hat{k}) \times (\hat{i} - \lambda\hat{j} + 7\hat{k}) = \vec{0}$. **[CBSE 2010]**

3. If $\vec{a} = (-3\hat{i} + 4\hat{j} - 7\hat{k})$ and $\vec{b} = (6\hat{i} + 2\hat{j} - 3\hat{k})$, find $(\vec{a} \times \vec{b})$.
   Verify that (i) $\vec{a}$ and $(\vec{a} \times \vec{b})$ are perpendicular to each other
   and (ii) $\vec{b}$ and $(\vec{a} \times \vec{b})$ are perpendicular to each other.

4. Find the value of:
   (i) $(\hat{i} \times \hat{j}) \cdot \hat{k} + \hat{i} \cdot \hat{j}$ (ii) $(\hat{k} \times \hat{j}) \cdot \hat{i} + \hat{j} \cdot \hat{k}$ **[CBSE 2012]**
   (iii) $\hat{i} \times (\hat{j} + \hat{k}) + \hat{j} \times (\hat{k} + \hat{i}) + \hat{k} \times (\hat{i} + \hat{j})$ **[CBSE 2014]**

5. Find the unit vectors perpendicular to both $\vec{a}$ and $\vec{b}$ when
   (i) $\vec{a} = 3\hat{i} + \hat{j} - 2\hat{k}$ and $\vec{b} = 2\hat{i} + 3\hat{j} - \hat{k}$
   (ii) $\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} - \hat{k}$
   (iii) $\vec{a} = \hat{i} + 3\hat{j} - 2\hat{k}$ and $\vec{b} = -\hat{i} + 3\hat{k}$
   (iv) $\vec{a} = 4\hat{i} + 2\hat{j} - \hat{k}$ and $\vec{b} = \hat{i} + 4\hat{j} - \hat{k}$.

6. Find the unit vectors perpendicular to the plane of the vectors $\vec{a} = 2\hat{i} - 6\hat{j} - 3\hat{k}$ and $\vec{b} = 4\hat{i} + 3\hat{j} - \hat{k}$.

7. Find a vector of magnitude 6 which is perpendicular to both the vectors $\vec{a} = 4\hat{i} - \hat{j} + 3\hat{k}$ and $\vec{b} = -2\hat{i} + \hat{j} - 2\hat{k}$.

8. Find a vector of magnitude 5 units, perpendiuclar to each of the vectors $(\vec{a} + \vec{b})$ and $(\vec{a} - \vec{b})$, where $\vec{a} = (\hat{i} + \hat{j} + \hat{k})$ and $\vec{b} = (\hat{i} + 2\hat{j} + 3\hat{k})$.
   [CBSE 2008C]

9. Find the angle between two vectors $\vec{a}$ and $\vec{b}$ with magnitudes 1 and 2 respectively and $|\vec{a} \times \vec{b}| = \sqrt{3}$. **[CBSE 2009]**

10. If $\vec{a}=(\hat{i}-\hat{j})$, $\vec{b}=(3\hat{j}-\hat{k})$ and $\vec{c}=(7\hat{i}-\hat{k})$, find a vector $\vec{d}$ which is perpendicular to both $\vec{a}$ and $\vec{b}$ and for which $\vec{c}\cdot\vec{d}=1$. **[CBSE 2010]**

11. If $\vec{a}=(4\hat{i}+5\hat{j}-\hat{k})$, $\vec{b}=(\hat{i}-4\hat{j}+5\hat{k})$, and $\vec{c}=(3\hat{i}+\hat{j}-\hat{k})$, find a vector $\vec{d}$ which is perpendicular to both $\vec{a}$ and $\vec{b}$ and for which $\vec{c}\cdot\vec{d}=21$.

12. Prove that $|\vec{a}\times\vec{b}|=(\vec{a}\cdot\vec{b})\tan\theta$, where θ is the angle between $\vec{a}$ and $\vec{b}$.

13. Write the value of p for which $\vec{a}=(3\hat{i}+2\hat{j}+9\hat{k})$ and $\vec{b}=(\hat{i}+p\hat{j}+3\hat{k})$ are parallel vectors. **[CBSE 2009]**

14. Verify that $\vec{a}\times(\vec{b}+\vec{c})=(\vec{a}\times\vec{b})+(\vec{a}\times\vec{c})$, when
    (i) $\vec{a}=\hat{i}-\hat{j}-3\hat{k}$, $\vec{b}=4\hat{i}-3\hat{j}+\hat{k}$ and $\vec{c}=2\hat{i}-\hat{j}+2\hat{k}$
    (ii) $\vec{a}=4\hat{i}-\hat{j}+\hat{k}$, $\vec{b}=\hat{i}+\hat{j}+\hat{k}$ and $\vec{c}=\hat{i}-\hat{j}+\hat{k}$.

15. Find the area of the parallelogram whose adjacent sides are represented by the vectors
    (i) $\vec{a}=\hat{i}+2\hat{j}+3\hat{k}$ and $\vec{b}=-3\hat{i}-2\hat{j}+\hat{k}$
    (ii) $\vec{a}=(3\hat{i}+\hat{j}+4\hat{k})$ and $\vec{b}=(\hat{i}-\hat{j}+\hat{k})$
    (iii) $\vec{a}=2\hat{i}+\hat{j}+3\hat{k}$ and $\vec{b}=\hat{i}-\hat{j}$
    (iv) $\vec{a}=2\hat{i}$ and $\vec{b}=3\hat{j}$.

16. Find the area of the parallelogram whose diagonals are represented by the vectors
    (i) $\vec{d}_1=3\hat{i}+\hat{j}-2\hat{k}$ and $\vec{d}_2=\hat{i}-3\hat{j}+4\hat{k}$ **[CBSE 2004]**
    (ii) $\vec{d}_1=2\hat{i}-\hat{j}+\hat{k}$ and $\vec{d}_2=3\hat{i}+4\hat{j}-\hat{k}$
    (iii) $\vec{d}_1=\hat{i}-3\hat{j}+2\hat{k}$ and $\vec{d}_2=-\hat{i}+2\hat{j}$.

17. Find the area of the triangle whose two adjacent sides are determined by the vectors
    (i) $\vec{a}=-2\hat{i}-5\hat{k}$ and $\vec{b}=\hat{i}-2\hat{j}-\hat{k}$
    (ii) $\vec{a}=3\hat{i}+4\hat{j}$ and $\vec{b}=-5\hat{i}+7\hat{j}$.

18. Using vectors, find the area of $\triangle ABC$ whose vertices are
    (i) $A(1, 1, 2)$, $B(2, 3, 5)$ and $C(1, 5, 5)$ **[CBSE 2011]**
    (ii) $A(1, 2, 3)$, $B(2, -1, 4)$ and $C(4, 5, -1)$ **[CBSE 2013]**
    (iii) $A(3, -1, 2)$, $B(1, -1, -3)$ and $C(4, -3, 1)$
    (iv) $A(1, -1, 2)$, $B(2, 1, -1)$ and $C(3, -1, 2)$.

19. Using vector method, show that the given points A, B, C are collinear:
    (i) $A(3, -5, 1)$, $B(-1, 0, 8)$ and $C(7, -10, -6)$
    (ii) $A(6, -7, -1)$, $B(2, -3, 1)$ and $C(4, -5, 0)$.

20. Show that the points A, B, C with position vectors $(3\hat{i} - 2\hat{j} + 4\hat{k})$, $(\hat{i} + \hat{j} + \hat{k})$ and $(-\hat{i} + 4\hat{j} - 2\hat{k})$ respectively are collinear.

21. Show that the points having position vectors $\vec{a}$, $\vec{b}$, $(\vec{c} = 3\vec{a} - 2\vec{b})$ are collinear, whatever be $\vec{a}$, $\vec{b}$, $\vec{c}$.

22. Show that the points having position vectors $(-2\vec{a} + 3\vec{b} + 5\vec{c})$, $(\vec{a} + 2\vec{b} + 3\vec{c})$ and $(7\vec{a} - \vec{c})$ are collinear, whatever be $\vec{a}$, $\vec{b}$, $\vec{c}$.

23. Find a unit vector perpendicular to the plane ABC, where the points A, B, C are $(3, -1, 2), (1, -1, -3)$ and $(4, -3, 1)$ respectively.

24. If $\vec{a} = (\hat{i} - 2\hat{j} + 3\hat{k})$ and $\vec{b} = (\hat{i} - 3\hat{k})$ then find $|\vec{b} \times 2\vec{a}|$.

25. If $|\vec{a}| = 2$, $|\vec{b}| = 5$ and $|\vec{a} \times \vec{b}| = 8$, find $\vec{a} \cdot \vec{b}$.

26. If $|\vec{a}| = 2$, $|\vec{b}| = 7$ and $(\vec{a} \times \vec{b}) = (3\hat{i} + 2\hat{j} + 6\hat{k})$, find the angle between $\vec{a}$ and $\vec{b}$.

ANSWERS (EXERCISE 24)

1. (i) $\vec{a} \times \vec{b} = (-2\hat{i} + 8\hat{j} + 5\hat{k})$ and $|\vec{a} \times \vec{b}| = \sqrt{93}$

(ii) $\vec{a} \times \vec{b} = (-17\hat{i} + 13\hat{j} + 7\hat{k})$ and $|\vec{a} \times \vec{b}| = 13\sqrt{3}$

(iii) $\vec{a} \times \vec{b} = 19(\hat{j} + \hat{k})$ and $|\vec{a} \times \vec{b}| = 19\sqrt{2}$

(iv) $\vec{a} \times \vec{b} = (\hat{i} - 10\hat{j} - 3\hat{k})$ and $|\vec{a} \times \vec{b}| = \sqrt{110}$

(v) $\vec{a} \times \vec{b} = (4\hat{i} - 3\hat{j} - \hat{k})$ and $|\vec{a} \times \vec{b}| = \sqrt{26}$

2. $\lambda = -3$ **3.** $(2\hat{i} - 51\hat{j} - 30\hat{k})$ **4.** (i) 1 (ii) –1 (iii) 0

5. (i) $\pm \frac{1}{5\sqrt{3}}(5\hat{i} - \hat{j} + 7\hat{k})$ (ii) $\pm \frac{1}{\sqrt{3}}(-\hat{i} + \hat{j} + \hat{k})$

(iii) $\pm \frac{1}{\sqrt{91}}(9\hat{i} - \hat{j} + 3\hat{k})$ (iv) $\pm \frac{1}{\sqrt{209}}(2\hat{i} + 3\hat{j} + 14\hat{k})$

6. $\pm\frac{1}{7}(3\hat{i}-2\hat{j}+6\hat{k})$ **7.** $\pm 2(-\hat{i}+2\hat{j}+2\hat{k})$ **8.** $\pm\frac{5(-\hat{i}+2\hat{j}-\hat{k})}{\sqrt{6}}$

9. $\frac{\pi}{3}$ **10.** $\vec{d}=\frac{1}{4}(\hat{i}+\hat{j}+3\hat{k})$ **11.** $\vec{d}=7(\hat{i}-\hat{j}-\hat{k})$ **13.** $p=\frac{2}{3}$

15. (i) $6\sqrt{5}$ sq units (ii) $\sqrt{42}$ sq units
(iii) $3\sqrt{3}$ sq units (iv) 6 sq units

16. (i) $5\sqrt{3}$ sq units (ii) $\frac{1}{2}\sqrt{155}$ sq units (iii) $\frac{1}{2}\sqrt{21}$ sq units

17. (i) $\frac{1}{2}\sqrt{165}$ sq units (ii) $\frac{41}{2}$ sq units

18. (i) $\frac{1}{2}\sqrt{61}$ sq units (ii) $\frac{1}{2}\sqrt{274}$ sq units (iii) $\frac{1}{2}\sqrt{165}$ sq units
(iv) $\sqrt{13}$ sq units

23. $\frac{-\hat{i}-2\hat{j}+4\hat{k}}{\sqrt{21}}$ **24.** $4\sqrt{29}$ **25.** 6 **26.** $\frac{\pi}{6}$

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 24)

4. (i) $(\hat{i}\times\hat{j})\cdot\hat{k}+\hat{i}\cdot\hat{j}=\hat{k}\cdot\hat{k}+0=(1+0)=1$ [$\because$ $\hat{i}\times\hat{j}=\hat{k}$ and $\hat{i}\cdot\hat{j}=0$].

(ii) $(\hat{k}\times\hat{j})\cdot\hat{i}+\hat{j}\cdot\hat{k}=-\hat{i}\cdot\hat{i}+0=(-1+0)=-1$.

5. Required vector $=\frac{\pm(\vec{a}\times\vec{b})}{|\vec{a}\times\vec{b}|}$.

6. Required vector $=\frac{\pm(\vec{a}\times\vec{b})}{|\vec{a}\times\vec{b}|}$.

7. Required vector $=\frac{\pm 6(\vec{a}\times\vec{b})}{|\vec{a}\times\vec{b}|}$.

8. Required vector $=\frac{\pm 5[(\vec{a}+\vec{b})\times(\vec{a}-\vec{b})]}{|(\vec{a}+\vec{b})\times(\vec{a}-\vec{b})|}$.

9. $|\vec{a}\times\vec{b}|=|\vec{a}||\vec{b}|\sin\theta=(1\times 2\times\sin\theta)$

$\therefore\ 2\sin\theta=\sqrt{3}\ \Rightarrow\ \sin\theta=\frac{\sqrt{3}}{2}\ \Rightarrow\ \theta=\frac{\pi}{3}$.

10. Since $\vec{d}$ is perpendicular to both $\vec{a}$ and $\vec{b}$, so it is parallel to $(\vec{a}\times\vec{b})$.

$\therefore\quad \vec{d}=\lambda(\vec{a}\times\vec{b})=\lambda(\hat{i}+\hat{j}+3\hat{k})=(\lambda\hat{i}+\lambda\hat{j}+3\lambda\hat{k})$

$$\vec{c}\cdot\vec{d}=1 \Rightarrow (7\hat{i}-\hat{k})\cdot(\lambda\hat{i}+\lambda\hat{j}+3\lambda\hat{k})=1$$

$$\Rightarrow 7\lambda-3\lambda=1 \Rightarrow 4\lambda=1 \Rightarrow \lambda=\frac{1}{4}.$$

$$\therefore \quad \vec{d}=\frac{1}{4}(\hat{i}+\hat{j}+3\hat{k}).$$

12. $(\vec{a}\times\vec{b})=|\vec{a}||\vec{b}|\sin\theta\cdot\hat{n}$

$\Rightarrow |\vec{a}\times\vec{b}|=ab\sin\theta$, where $|\vec{a}|=a$ and $|\vec{b}|=b$

$$=(ab\cos\theta)(\tan\theta)=(\vec{a}\cdot\vec{b})\tan\theta.$$

13. $\vec{a}$ and $\vec{b}$ are parallel vectors $\Leftrightarrow \vec{a}\times\vec{b}=\vec{0}$.

15. (iv) Area $=|\vec{a}\times\vec{b}|=|2\hat{i}\times 3\hat{j}|=|6(\hat{i}\times\hat{j})|=|6\hat{k}|=6 \quad [\because (\hat{i}\times\hat{j})=\hat{k}]$.

16. Area of a $\|gm=\frac{1}{2}|\vec{d}_1\times\vec{d}_2|$.

17. Area of a triangle $=\frac{1}{2}|\vec{a}\times\vec{b}|$, where $\vec{a}$ and $\vec{b}$ are adjacent sides.

23. Required vector $=\dfrac{(\overrightarrow{AB}\times\overrightarrow{AC})}{|\overrightarrow{AB}\times\overrightarrow{AC}|}$.

25. $|\vec{a}\times\vec{b}|=|\vec{a}||\vec{b}|\sin\theta \Rightarrow 2\times5\times\sin\theta=8 \Rightarrow \sin\theta=\frac{4}{5}$

$$\Rightarrow \cos\theta=\sqrt{1-\frac{16}{25}}=\sqrt{\frac{9}{25}}=\frac{3}{5}.$$

$$\therefore \quad \vec{a}\cdot\vec{b}=|\vec{a}||\vec{b}|\cos\theta=\left(2\times5\times\frac{3}{5}\right)=6.$$

26. $\vec{a}\times\vec{b}=(3\hat{i}+2\hat{j}+6\hat{k})$

$\Rightarrow |\vec{a}\times\vec{b}|=\sqrt{3^2+2^2+6^2}=7 \Rightarrow |\vec{a}||\vec{b}|\sin\theta=7$

$$\Rightarrow \sin\theta=\frac{7}{|\vec{a}||\vec{b}|}=\frac{7}{(2\times7)}=\frac{1}{2} \Rightarrow \theta=30°.$$